

Characteristic classes as obstructions to local homogeneity

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1 Summary

Let M be a smooth manifold with $\dim M = n$ and $\{U_\alpha\}$ be an atlas with transition functions $\phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}^n$. Are there any "global invariants" of M (at least for certain M) which depend on the k -th order derivatives of $\phi_\beta \circ \phi_\alpha^{-1}$ for arbitrarily large k (as M varies)? Equivalently, do the higher order derivatives play any role in global differential geometry? This note is the outcome of our efforts over a period of 20 years and gives, we hope, an affirmative answer to this question in Section 7. Our method produces also obstructions to k -flatness as defined in [9]. We will shortly outline here the construction of these invariants which turn out to be the "old friends" but seen with a new eye.

A prehomogeneous geometry $\varepsilon\mathcal{G}_k$ (*phg* for short) of order k on M is a very special transitive Lie groupoid on M . The integer $k \geq 0$ indicates the order of jets involved in the definition of $\varepsilon\mathcal{G}_k$. Now $\varepsilon\mathcal{G}_0$ is an absolute parallelism on M , $k = 1$ for Riemannian geometry but can be arbitrarily large for parabolic geometries (like projective and conformal geometries) *as defined* in Section 2. The curvature \mathcal{R}_k of $\varepsilon\mathcal{G}_k$ vanishes if and only if the *PDE* defined by $\varepsilon\mathcal{G}_k$ is locally solvable. In geometric terms, this is equivalent to the local homogeneity of M in the way imposed by $\varepsilon\mathcal{G}_k$. With the assumption of completeness and simple connectedness, M becomes a globally homogeneous space G/H possibly with noncompact H . In fact, compactness of H forces $k \leq 1$. We have $k \leq \dim \text{Nil}(\mathfrak{h}) + 1$ so that $k \leq 1$ also if H is semisimple. However $\varepsilon\mathcal{G}_k$ is not modeled on some fixed G/H chosen beforehand.

The algebroid $\varepsilon\mathfrak{G}_k \rightarrow M$ of $\varepsilon\mathcal{G}_k$ is a very special vector bundle filtered by jets. The Chern-Weil construction applied to the curvature \mathfrak{R}_k of $\varepsilon\mathfrak{G}_k \rightarrow M$ establishes the Pontryagin algebra $\mathcal{P}^*(M, \varepsilon\mathfrak{G}_k) \subset H_{dR}^*(M, \mathbb{R})$ as an obstruction to local homogeneity. In other words, the well known characteristic classes of vector bundles become obstructions to integrability once they are restricted to this particular subset of vector bundles. These obstructions depend on first order jets and are topological. Using the projections $\mathcal{G}_k \rightarrow \mathcal{G}_r$, $0 \leq r \leq k$, we define the higher order Pontryagin algebras $\mathcal{P}^*(\mathcal{G}_r^\bullet, \varepsilon\mathfrak{G}_k) \subset H^*(\mathcal{G}_r^\bullet, \mathbb{R})$ where $\mathcal{G}_r^\bullet \rightarrow M$ is the principal bundle of the groupoid \mathcal{G}_r so that $\mathcal{G}_0 = M \times M$, $\mathcal{G}_0^\bullet = M$. For $1 \leq r \leq k$, $\mathcal{P}^*(\mathcal{G}_r^\bullet, \varepsilon\mathfrak{G}_k)$ is trivial as a subalgebra of $H_{dR}^*(\mathcal{G}_r^\bullet, \mathbb{R})$ due to the contractibility of the fibers of $\mathcal{G}_k^\bullet \rightarrow \mathcal{G}_r^\bullet$. However, the representatives of $\mathcal{P}^*(\mathcal{G}_r^\bullet, \varepsilon\mathfrak{G}_k)$ are right invariant forms on the principal bundle $\mathcal{G}_r^\bullet \rightarrow M$ and generate a subalgebra $\widehat{\mathcal{P}}^*(\mathcal{G}_r^\bullet, M)$ in the subcomplex of right invariant forms which computes the algebroid cohomology of $\mathcal{G}_r^\bullet \rightarrow M$. The subalgebra $\widehat{\mathcal{P}}^*(\mathcal{G}_r^\bullet, M) \subset H_{inv}^*(\mathcal{G}_r^\bullet, M) = H^*(M, \mathfrak{G}_r)$, we believe, need not be trivial for $1 \leq r \leq k$ and gives obstructions to local homogeneity which depend on jets of order k . All these obstructions depend on the isomorphism class $[\varepsilon\mathcal{G}_k]$ of $\varepsilon\mathcal{G}_k$. In view of the definition of $[\varepsilon\mathcal{G}_k]$, the assignment $[\varepsilon\mathcal{G}_k] \Rightarrow \mathcal{P}^*(\mathcal{G}_r^\bullet, \varepsilon\mathfrak{G}_k)$ is tantamount to the assignment of certain invariants to the moduli space of connections on the principal bundle $\mathcal{G}_k^\bullet \rightarrow M$ as in gauge theory. In Section 8 we interpret the above Pontryagin algebras as obstructions to the existence of certain Cartan connections. In Section 9 we show that the Chern-Simons forms arise naturally in the present framework but with a surprisingly different interpretation.

2 Prehomogeneous geometries

Let M be a smooth manifold with $\dim M = n \geq 2$ and $j_k(f)^{p,q}$ be the k -jet of the local diffeomorphism f with source at p and target at q . We call $j_k(f)^{p,q}$ a k -arrow from p to q . Clearly $j_0(f)^{p,q} = (p, q)$. Let $\mathcal{U}_k^{p,q}$ denote the set of all k -arrows from p to q . With the composition and inversion of k -arrows, the set $\mathcal{U}_k \stackrel{\text{def}}{=} \cup_{p,q \in M} \mathcal{U}_k^{p,q}$ of all k -arrows on M has the structure of a groupoid

which we call the universal k -th order groupoid on M . The set $\mathcal{U}_k^{p,p}$, $p \in M$, is a Lie group and called the vertex group of \mathcal{U}_k at p . A choice of coordinates around p identifies $\mathcal{U}_k^{p,p}$ with the k -th order jet group $G_k(n)$ in n variables. We define $G_0(n)$ as the set containing one point. The projection of jets induces a homomorphism $\pi_{k+1,k} : \mathcal{U}_{k+1} \rightarrow \mathcal{U}_k$ of groupoids and we have the sequence of projections

$$\dots \rightarrow \mathcal{U}_{k+1} \rightarrow \mathcal{U}_k \rightarrow \dots \rightarrow \mathcal{U}_1 \rightarrow \mathcal{U}_0 = M \times M \quad (1)$$

Note that (1) can be restricted to the vertex groups at p . The set $\mathcal{U}_k^{e,\bullet} \stackrel{def}{=} \cup_{x \in M} \mathcal{U}_k^{e,x}$ is a principal bundle with the structure group $\mathcal{U}_k^{e,e} \simeq G_k(n)$ where $e \in M$ is some basepoint and (1) restricts also to these principal bundles.

In this note we will be interested in certain subgroupoids $\mathcal{G}_k \subset \mathcal{U}_k$. For $s \leq k$, we denote $\pi_{k,s} \mathcal{G}_k$ by $\mathcal{G}_s \subset \mathcal{U}_s$.

Definition 1 *A prehomogeneous geometry (phg) of order k on M is a subgroupoid $\mathcal{G}_{k+1} \subset \mathcal{U}_{k+1}$ satisfying*

- i) $\mathcal{G}_0 = \mathcal{U}_0 = M \times M$*
- ii) $\mathcal{G}_{k+1} \simeq \mathcal{G}_k$ and k is the smallest such integer*

So $\mathcal{G}_k \subset \mathcal{U}_k$ is an imbedded submanifold consisting of certain k -arrows of \mathcal{U}_k closed under composition and inversion. *i)* states that \mathcal{G}_k is transitive on M , i.e., for any $p, q \in M$ there exists a k -arrow of \mathcal{G}_k from p to q . Let $\varepsilon : \mathcal{G}_k \rightarrow \mathcal{G}_{k+1}$ denote the inverse of the isomorphism given by *ii)* so that $\mathcal{G}_{k+1} = \varepsilon \mathcal{G}_k$. Now *ii)* states that above any k -arrow $j_k(f)^{p,q}$ in $\mathcal{G}_k^{p,q}$ there exists a unique $(k+1)$ -arrow (namely $\varepsilon j_k(f)^{p,q}$) and this 1-1 correspondence preserves composition and inversion of arrows since ε is an isomorphism of groupoids. The second condition in *ii)* states that \mathcal{G}_r projects onto \mathcal{G}_s with nontrivial kernel for $1 \leq s+1 \leq r \leq k$. Since \mathcal{G}_k is transitive, this condition holds if and only if it holds at one (hence all) vertex group. Note that ε restricts to the vertex groups $\mathcal{G}_k^{p,p}$ and also to the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$. As we will see shortly, many (if not all) geometric structures are particular *phg*'s. We will add a third condition *iii)* to Definition 1 below when it is needed.

Choosing coordinates (U, x^i) , (V, y^i) , elements of $\mathcal{U}_{k+1}^{p,q}$ with $p \in U$, $q \in V$ can be expressed locally as $(x^i, y^i, f_{j_1}^i, f_{j_2 j_1}^i, \dots, f_{j_{k+1} \dots j_1}^i)$. Since $\varepsilon \mathcal{G}_k \subset \mathcal{U}_{k+1}$ is a submanifold, locally it is defined by a set of independent equations

$$\Phi^\alpha(x^i, y^i, f_{j_1}^i, f_{j_2 j_1}^i, \dots, f_{j_{k+1} \dots j_1}^i) = 0 \quad 1 \leq \alpha \leq \dim \mathcal{U}_{k+1} - \dim \varepsilon \mathcal{G}_k \quad (2)$$

The functions Φ^α are surely not unique and the study of their invariance properties gives rise to a subtle local theory which we will not touch here. Note that (2) puts no restriction on the variables x^i, y^i by *i)*. Since $f_{j_{k+1} \dots j_1}^i$ is determined by $x^i, y^i, f_{j_1}^i, \dots, f_{j_k \dots j_1}^i$ by *ii)*, we can solve $f_{j_{k+1} \dots j_1}^i$ in terms of $x^i, y^i, f_{j_1}^i, \dots, f_{j_k \dots j_1}^i$ and rewrite (2) in an equivalent form.

We now fix an "initial condition" $(\bar{x}^i, \bar{y}^i, \bar{f}_{j_1}^i, \bar{f}_{j_2 j_1}^i, \dots, \bar{f}_{j_{k+1} \dots j_1}^i)$ satisfying (2). We search for a diffeomorphism $f : U \rightarrow f(U) \subset V$ such that $(x^i, f^i(x), \frac{\partial f^i(x)}{\partial x^{j_1}}, \dots, \frac{\partial f^i(x)}{\partial x^{j_{k+1}}})$,

..., $\frac{\partial^k f^i(x)}{\partial x^{j_{k+1}} \dots \partial x^{j_1}}$) solves (2) for all $x \in U$ and also satisfies $f^i(\bar{x}) = \bar{y}^i$, $\frac{\partial f^i(\bar{x})}{\partial x^{j_1}} = \bar{f}_{j_1}^i, \dots, \frac{\partial^k f^i(\bar{x})}{\partial x^{j_{k+1}} \dots \partial x^{j_1}} = \bar{f}_{j_{k+1} \dots j_1}^i$. This interpretation shows that $\varepsilon\mathcal{G}_k$ is a nonlinear *PDE* of order $k+1$ defined on the universal pseudogroup $Diff_{loc}(M)$ of all local diffeomorphisms of M and is locally of the form (2). The $(k+1)$ -arrows of $\varepsilon\mathcal{G}_k$ are the initial conditions. In a coordinate free language, let $\alpha_{k+1}^{p,q} \in \varepsilon\mathcal{G}_k$. A local diffeomorphism $f : U \rightarrow f(U)$ with $f(p) = q$ is a local solution of $\varepsilon\mathcal{G}_k$ on U satisfying the initial condition $\alpha_{k+1}^{p,q}$ if

- i) $j_{k+1}(f)^{p,q} = \alpha_{k+1}^{p,q}$
- ii) $j_{k+1}(f)^{x,f(x)} \in \varepsilon\mathcal{G}_k$ for all $x \in U$

A local solution, if it exists, satisfies all its $(k+1)$ -arrows as initial conditions.

Proposition 2 *If f, g are two local solutions with $j_{k+1}(f)^{p,q} = j_{k+1}(g)^{p,q}$, then $f = g$ on their common domain of definition.*

Proposition 2 states that local solutions, if they exist, are unique. This can be seen roughly by noting that ε expresses jets of order $k+1$ in terms of the lower order jets so that the Taylor expansion of a local solution satisfying some initial condition is determined by this initial condition.

Definition 3 *$\varepsilon\mathcal{G}_k$ is locally solvable if all its $(k+1)$ -arrows integrate to local solutions as above.*

Suppose $\varepsilon\mathcal{G}_k$ is locally solvable and let $\widetilde{\varepsilon\mathcal{G}_k}$ denote the set of all local diffeomorphism obtained by integrating the $(k+1)$ -arrows of $\varepsilon\mathcal{G}_k$. Since $\varepsilon\mathcal{G}_k$ is a groupoid, we easily see that $\widetilde{\varepsilon\mathcal{G}_k}$ is a pseudogroup and we say that $\varepsilon\mathcal{G}_k$ integrates to $\widetilde{\varepsilon\mathcal{G}_k}$. Therefore, if $\varepsilon\mathcal{G}_k$ is locally solvable, then M is locally homogeneous in the way imposed by $\varepsilon\mathcal{G}_k$.

Now suppose that $\varepsilon\mathcal{G}_k$ is locally solvable. Let $f \in \widetilde{\varepsilon\mathcal{G}_k}$ with $f(p) = q$ and γ be a (continuous) path from p to some point r . Using Proposition 2 we can "analytically continue" $j_{k+1}(f)^{p,q}$ along this path but we may not be able to end up with a $(k+1)$ -arrow with source at r .

Definition 4 *If elements of $\widetilde{\varepsilon\mathcal{G}_k}$ can be analytically continued indefinitely along paths, then $\varepsilon\mathcal{G}_k$ is complete.*

If $f \in \widetilde{\varepsilon\mathcal{G}_k}$ is the restriction of some (unique!) global transformation $\tilde{f} \in Diff(M)$, we call f globalizable.

Definition 5 *$\widetilde{\varepsilon\mathcal{G}_k}$ is globalizable if all $f \in \widetilde{\varepsilon\mathcal{G}_k}$ are globalizable.*

Hence if $\widetilde{\varepsilon\mathcal{G}_k}$ is globalizable then we obtain a global transformation group G which acts transitively and effectively on M . We call this data a Klein geometry (G, M) which we can identify with the homogeneous space $G/H = M$ where H is the stabilizer at some point. Note that this identification is not canonical and depends on the choice of a base point. Obviously $\widetilde{\varepsilon\mathcal{G}_k}$ is complete if it

is globalizable. Conversely, let $f \in \widetilde{\varepsilon\mathcal{G}_k}$ with $f(p) = q$. Assuming that $\widetilde{\varepsilon\mathcal{G}_k}$ is complete and M is simply connected, we define a map $\tilde{f} : M \rightarrow M$ as follows: for any $r \in M$, we choose a path from p to r , continue f along this path up to r and define $\tilde{f}(r)$ to be the value of this continuation. A standard monodromy argument using simple connectedness shows that $\tilde{f}(r)$ is independent of the path from p to r and we easily check that \tilde{f} is 1-1 and onto. Thus we have

Proposition 6 *If $\widetilde{\varepsilon\mathcal{G}_k}$ is complete and simply connected then it is globalizable.*

If $\widetilde{\varepsilon\mathcal{G}_k}$ is complete but not globalizable, then we can pull back $\widetilde{\varepsilon\mathcal{G}_k}$ to the universal covering space $\pi : \mathcal{M} \rightarrow M$. Since $\pi^*\widetilde{\varepsilon\mathcal{G}_k}$ is complete and \mathcal{M} is simply connected, $\pi^*\widetilde{\varepsilon\mathcal{G}_k}$ globalizes to a Lie group G acting on \mathcal{M} and we obtain the Klein geometry $(G, \mathcal{M}) \simeq G/H = \mathcal{M}$. To summarize, we have

Proposition 7 *Let $\varepsilon\mathcal{G}_k$ be locally solvable and complete. Then the pseudogroup $\widetilde{\varepsilon\mathcal{G}_k}$ globalizes to a Lie group G on the universal covering space \mathcal{M} so that $(G, \mathcal{M}) \simeq G/H = \mathcal{M}$ and $M = G/H \setminus \Gamma$ for some discrete subgroup $\Gamma \subset G$ which is isomorphic to the fundamental group of M .*

A pseudogroup arising from a locally solvable phg as above is a finite type Lie pseudogroup according to [14].

Observe that we defined completeness of $\varepsilon\mathcal{G}_k$ only when it is locally solvable. We will turn back to this issue in Section 5.

Conversely we now start with a transitive and effective Klein geometry $(G, M) \simeq G/H = M$. We assume that G is connected and M is simply connected (so H is also connected) for reasons which will be clear below. Note that (G, M) always lifts to a Klein geometry (\tilde{G}, \mathcal{M}) where \mathcal{M} is the universal cover of M .

We fix some base point $e \in M$ and let $H_e = \{g \in G \mid g(e) = e\}$. Recall that a coordinate system around e identifies $\mathcal{U}_k^{e,e}$ with the jet group $G_k(n)$. We have the evaluation maps

$$\begin{aligned} j_i^e & : H_e \longrightarrow \mathcal{U}_i^{e,e} \simeq G_i(n) \\ & : h \longrightarrow j_i(h)^{e,e} \quad 0 \leq i \end{aligned} \tag{3}$$

which are clearly homomorphisms of Lie groups. Since G is connected and (G, M) is effective (as we always assume in this note), there exists an integer k such that j_k^e becomes injective ([2]).

Definition 8 *The smallest integer k such that (3) becomes injective is the order of (G, M) denoted by $\text{ord}(G, M)$.*

Since G acts transitively, this integer is independent of our choice of the base point e . Clearly, $\text{ord}(G, M) = 0$ if and only if G acts simply transitively.

Definition 8 needs only connectedness of G . If M is not simply connected, then $\text{ord}(G, M)$ may be one greater than the one in Definition 8 and the length of the top filtration in (13) below may be one greater than the bottom filtration.

Now any $g \in G$ is determined globally by its k -arrow $j_k(g)^{p,q}$ for any $p, q \in M$. Indeed, $j_k(g)^{p,q} = j_k(g')^{p,q} \Leftrightarrow j_k(g' \circ g^{-1})^{p,p} = Id_k^{p,p} \Leftrightarrow g = g'$ since j_k^p is injective. We define the groupoid $\mathcal{G}_k \subset \mathcal{U}_k$ on M by defining its fiber $\mathcal{G}_k^{p,q} \stackrel{def}{=} \{j_k(f)^{p,q} \mid f \in G, f(p) = q\}$. Further, $j_k(f)^{p,q}$ determines f which in turn determines $j_{k+1}(f)^{p,q} \stackrel{def}{=} \varepsilon j_k(f)^{p,q}$. Thus we obtain a splitting ε such that $\mathcal{G}_k \simeq \varepsilon \mathcal{G}_k \subset \mathcal{U}_{k+1}$. However there is a technical difficulty: Even though the map j_k^e is smooth as it is continuous, the image $j_k(H_e) \subset \mathcal{U}_k^{e,e}$ need not be a closed subgroup and therefore the groupoid $\varepsilon \mathcal{G}_k \subset \mathcal{U}_{k+1}$ need not be a subgroupoid which should be an imbedded submanifold. Such an example is given in [19]. If H_e is compact, this anomaly can not occur but in this case $ord(G, M) \leq 1$ by Proposition 23 below so this is a very strong condition. We do not know any sufficient condition which makes $j_k(H_e) \subset \mathcal{U}_k^{e,e}$ closed but does not restrict $ord(G, M)$.

In this note we will make the overall assumption

A1: The injection

$$j_k^e : H_e \rightarrow \mathcal{U}_k^{e,e} \simeq G_k(n) \quad (4)$$

imbeds H_e as a closed subgroup for some (hence all) base point $e \in M$.

Clearly, $j_k(H_e) \subset \mathcal{U}_k^{e,e}$ is closed $\Leftrightarrow \varepsilon j_k(H_e) \subset \mathcal{U}_{k+1}^{e,e}$ is closed. Henceforth we will identify H_e with its image $\varepsilon j_k(H_e)$.

Therefore we deduce

Proposition 9 *A Klein geometry (G, M) determines a locally solvable (in fact globally solvable admitting G as its global solution space) phg $\varepsilon \mathcal{G}_k$ where $k = ord(G, M)$.*

To summarize what we have done so far, a locally solvable $\varepsilon \mathcal{G}_k$ makes M locally homogeneous. With the assumption of completeness, the universal cover \widetilde{M} becomes globally homogeneous. Conversely any Klein geometry (G, M) with $ord(G, M) = k$ determines a globally solvable $\varepsilon \mathcal{G}_k$ with the above assumptions.

Now suppose that the identification $\mathcal{U}_{k+1}^{e,e} \simeq G_{k+1}(n)$ in (4) is induced by some coordinates (U, x^i) around e . Since a change of coordinates $(x^i) \rightarrow (y^i)$ conjugates this identification, $H_e \simeq \varepsilon j_k(H_e)$ defines a unique conjugacy class inside $G_{k+1}(n)$. Since G acts transitively, this conjugacy class is also independent of the choice of the basepoint e . With an abuse of notation, we denote this conjugacy class by $\langle H \rangle_G$ where H stands for any stabilizer of (G, M) .

Definition 10 *The conjugacy class $\langle H \rangle_G$ inside $G_{k+1}(n)$ is the vertex class of $(G, M) \simeq G/H = M$.*

We now fix H , $\dim M$ and want to understand the dependence of $\langle H \rangle_G$ from G as (G, M) varies where G is connected and M is simply connected as we assumed above. The below examples show that we may have $\langle H \rangle_G = \langle H \rangle_{G'}$ but G, G' are not even locally isomorphic.

Example 1: Consider the projection $G_1(n) \rightarrow G_0(n) = \{1\}$ with the only splitting $\varepsilon(1) = 1$. Let $\langle \varepsilon 1 \rangle_{G_1(n)} = \{1\}$ denote the conjugacy class of $\varepsilon G_0(n)$

inside $G_1(n)$. Now *any* Klein geometry (G, M) with $\text{ord}(G, M) = 0$ defines this vertex class.

Example 2: Consider

$$0 \rightarrow K_{2,1} \rightarrow G_2^\circ(n) \rightarrow G_1^\circ(n) \rightarrow 1 \quad (5)$$

where $G_2^\circ(n)$, $G_1^\circ(n)$ are the connected components of $G_2(n)$, $G_1(n)$. Let $\varepsilon : G_1^\circ(n) \rightarrow G_2^\circ(n)$ be the splitting defined by $(a_j^i) \rightarrow (a_j^i, 0)$, i.e., $a_{jk}^i = 0$, $1 \leq i, j, k \leq n$. Let $\langle \varepsilon G_1^\circ(n) \rangle_{G_2^\circ(n)}$ denote conjugacy class of $\varepsilon G_1^\circ(n)$ inside $G_2^\circ(n)$ which we call the affine vertex class (of dimension n). Any other such splitting defines the same conjugacy class! The Klein geometry $G_1^\circ(n) \rtimes \mathbb{R}^n / G_1^\circ(n)$ has order one and

$$\langle G_1^\circ(n) \rangle_{G_1^\circ(n) \rtimes \mathbb{R}^n} = \langle \varepsilon G_1^\circ(n) \rangle_{G_2^\circ(n)} \quad (6)$$

Example 3: We restrict ε in (5) to the orthogonal group $SO(n)$ and let $\langle \varepsilon SO(n) \rangle_{G_2^\circ(n)}$ denote the conjugacy class of $\varepsilon(SO(n))$ inside $G_2^\circ(n)$.

Now consider the three Klein geometries $SO(n+1)/SO(n) = S^n$, $SO(n) \rtimes \mathbb{R}^n / SO(n) = \mathbb{R}^n$, $SO(n, 1)/SO(n) = \mathbb{H}^n$. These Klein geometries have order one and we have

$$\langle SO(n) \rangle_{SO(n+1)} = \langle SO(n) \rangle_{SO(n) \rtimes \mathbb{R}^n} = \langle SO(n) \rangle_{SO(n, 1)} = \langle \varepsilon SO(n) \rangle_{G_2^\circ(n)} \quad (7)$$

So three nonisomorphic Lie groups define the same vertex class.

Example 4: Consider

$$0 \rightarrow K_{3,2}(1) \rightarrow G_3^\circ(1) \rightarrow G_2^\circ(1) \rightarrow 1 \quad (8)$$

An element of $G_3(1)$ is an ordered triple (a_1, a_2, a_3) , $a_1 \neq 0$ and chain rule gives the group operation

$$(a_1, a_2, a_3)(b_1, b_2, b_3) = (a_1 b_1, a_1 b_2 + a_2 (b_1)^2, a_1 b_3 + 3a_2 b_1 b_2 + a_3 (b_1)^3) \quad (9)$$

We define $\varepsilon : G_2^\circ(1) \rightarrow G_3^\circ(1)$ by

$$\varepsilon(a_1, a_2) = (a_1, a_2, \frac{3(a_2)^2}{2a_1}) \quad (10)$$

Using (9) we check that ε is a homomorphism (and $(a_1, a_2, \varepsilon(a_1, a_2))^{-1}(a_1, a_2, a_3) = (1, 0, S(a_1, a_2, a_3))$ where S is the Schwarzian derivative!). Let $\langle \varepsilon G_2^\circ(1) \rangle_{G_3^\circ(1)}$ denote the conjugacy class of $\varepsilon G_2^\circ(1)$ inside $G_3^\circ(1)$.

Let \mathfrak{M} be the group of Mobius transformations $f(z) = \frac{az+b}{cz+d}$ normalized by $ad - bc = 1$ acting transitively and effectively on the sphere S^2 . Now $\text{ord}(\mathfrak{M}, S^2) = 2$ and $\langle H \rangle_{\mathfrak{M}} = \langle \varepsilon G_2^\circ(1) \rangle_{G_3^\circ(1)}$.

The above examples show that the dependence of $\langle H \rangle_G$ on G is quite subtle. However the problem can be reduced to algebra as follows. Let $(\mathfrak{g}, \mathfrak{h})$ be a pair of Lie algebras satisfying

- 1) $\mathfrak{h} \subset \mathfrak{g}$
- 2) $\dim \mathfrak{g} - \dim \mathfrak{h} = n$
- 3) $(\mathfrak{g}, \mathfrak{h})$ is effective, i.e., \mathfrak{h} contains no nontrivial ideals inside \mathfrak{g} .

Now we fix \mathfrak{h} and regard \mathfrak{g} as variable. For any two such pairs, we define $(\mathfrak{g}, \mathfrak{h}) \sim (\mathfrak{g}', \mathfrak{h})$ if $\mathfrak{g} \simeq \mathfrak{g}'$ and the isomorphism \simeq restricts to identity on \mathfrak{h} . The problem is to understand the equivalence classes. In Example 1 this is the formidable problem of classifying all Lie algebras whereas in Example 3 the solution is well known from Riemannian geometry. By fixing n and \mathfrak{h} , we call the cardinality of the equivalence classes the uniformization number $\#(\mathfrak{h}, n)$. So $\#(0, n) = \infty$, $\#(\mathfrak{aff}(n), n) = 1$ and $\#(\mathfrak{o}(n), n) = 3$, $n \geq 2$. We will comment more on $\#(\mathfrak{h}, n)$ in Appendix C.

We defined so far the vertex class $\langle H \rangle_G$ of $(G, M) \simeq G/H = M$. In the same way, we define the vertex class $\langle \varepsilon \mathcal{G}_k \rangle_{G_{k+1}(n)}$ of any $phg \in \mathcal{G}_k$ as the conjugacy class of $\varepsilon \mathcal{G}_{k+1}^{p,p} \subset \mathcal{U}_{k+1}^{p,p} \simeq G_{k+1}(n)$ inside $G_{k+1}(n)$. Like $\langle H \rangle_G$, $\langle \varepsilon \mathcal{G}_k \rangle_{G_{k+1}(n)}$ does not depend on the identification $\mathcal{U}_{k+1}^{p,p} \simeq G_{k+1}(n)$ induced by some coordinates around p and is also independent of the choice of p by transitivity.

In the above examples we started with the conjugacy class of some subgroup εH inside some jet group and exhibited some Klein geometries (∞ , 1 and 3 in number) with the vertex classes equal to the conjugacy class of εH . Is this possible for any such εH ? So we face the following question

Q: For some arbitrary phg , does there exist some G/H with $\langle \varepsilon \mathcal{G}_k \rangle_{G_{k+1}(n)} = \langle H \rangle_G$?

We do not know the answer. Therefore we add the third condition *iii*) to the Definition 1:

iii) There exists a Klein geometry G/H with $\langle \varepsilon \mathcal{G}_k \rangle_{G_{k+1}(n)} = \langle H \rangle_G$.

iii) is important for the following reason. We can restrict $\varepsilon \mathcal{G}_k$ to any open subset $U \subset M$ and the restriction $\varepsilon \mathcal{G}_{k|U}$ also satisfies *i*), *ii*). In Sections 6,7 we will assign certain invariants to phg 's which depend on their equivalence class and vanish if there is a locally solvable phg in this equivalence class. We want these invariants vanish for $\varepsilon \mathcal{G}_{k|U}$ for any $\varepsilon \mathcal{G}_k$ if U is sufficiently small. This will be the case if $\varepsilon \mathcal{G}_{k|U}$ is equivalent to some phg on U which is locally solvable and *iii*) implies this. In short, we want a phg to be locally equivalent to a locally solvable one.

According to *iii*), we now state

$$\text{A } phg \text{ is modeled on some } \langle H \rangle_G \text{ defined by some } (G, M) \simeq G/H \quad (11)$$

We now give some examples of phg 's on some (not necessarily simply connected) M .

Example 1 (continues): For $k = 0$, $\mathcal{G}_0 = M \times M$ and ε assigns to any pair (p, q) a unique 1-arrow from p to q . So M admits $\varepsilon \mathcal{G}_0$ if and only if it is parallelizable. Clearly, $\langle \varepsilon \mathcal{G}_0 \rangle = \langle 1 \rangle_G$ for any G with $\dim G = \dim M$. If $\varepsilon \mathcal{G}_0$ is locally solvable, we get the pseudogroup $\widetilde{\varepsilon \mathcal{G}_0}$ on M which acts simply transitively and, assuming completeness, globalizes to some Lie group G on the universal cover of M . Any Lie group G is a possibility. Therefore, the case $k = 0$

gives the theory of parallelizable manifolds and Lie groups as simply transitive transformation groups. In Section 10 we will take a more careful look at this case.

Example 2 (continues): We recall \mathcal{U}_1 and let ε be any symmetric connection on $T \rightarrow M$. The transformation rule of the components $\left(\varepsilon_{jk}^i\right)$ shows that ε defines above any 1-arrow of \mathcal{U}_1 a unique 2-arrow of \mathcal{U}_2 . The *phg* $\varepsilon\mathcal{U}_1$ has order one and $\langle \varepsilon\mathcal{U}_1 \rangle = \langle \varepsilon G_1^\circ(n) \rangle_{G_2(n)}$.

Definition 11 $\varepsilon\mathcal{U}_1$ is an *affine geometry* on M .

The curvature \mathcal{R}_1 of $\varepsilon\mathcal{U}_1$, as defined in Section 3, is *not the same object* as the well known curvature \mathcal{R} of ε ! Indeed \mathcal{R} is a tensor whereas \mathcal{R}_1 is a second order object! However $\mathcal{R}_1 = 0 \Leftrightarrow \mathcal{R} = 0$. In this case, assuming completeness, the pseudogroup $\widetilde{\varepsilon\mathcal{U}_1}$ globalizes to $Aff^\circ(\mathbb{R}^n) = G_1^\circ(n) \rtimes \mathbb{R}^n$ on the universal cover $\mathcal{M} \simeq \mathbb{R}^n$ of M . There is no possibility other than $(Aff^\circ(\mathbb{R}^n), \mathbb{R}^n)$.

Example 3 (continues): Let g be a metric on M . We define $\mathcal{G}_1^{p,q}$ as the set of all 1-arrows from p to q which map $g(p)$ to $g(q)$. The transformation rule of the Christoffel symbols ε_{jk}^i shows that above any such 1-arrow, there is a unique 2-arrow which defines $\varepsilon\mathcal{G}_1 \subset \mathcal{U}_2$. To be consistent with above general philosophy, we also assume that the elements in the vertex groups have positive determinant.

Definition 12 $\varepsilon\mathcal{G}_1$ is a "*Riemann geometry*" on M

Observe that a "Riemann geometry" according to Definition 12 is a second order structure but not a first order structure! Now $\varepsilon\mathcal{G}_1$ is locally solvable $\Leftrightarrow g$ has constant curvature. In particular \mathcal{R}_1 is *not the Riemann curvature tensor but a second order geometric object!!* A. Blaom gives a very simple and explicit formula for \mathcal{R}_1 on pg. 6 of [4]. If $\varepsilon\mathcal{G}_1$ is locally solvable, the pseudogroup $\widetilde{\varepsilon\mathcal{G}_1}$ globalizes, assuming completeness, on the universal cover of M to one of the three groups in (7). There are no other possibilities other than these three groups.

Example 4 (continues): As a generalization of Example 4, we now want to define a projective geometry.

We first observe $\mathfrak{M} \simeq SL(2, \mathbb{R})$ $H \simeq$ the upper triangular matrices = the stabilizer of ∞ obtained by setting $c = 0$.

We will denote H by $B(2)$. Now now fix $B(n) \subset SL(n, \mathbb{R})$ as our stabilizer. However we are not forced to fix $SL(n, \mathbb{R})$ because a *phg* is not modeled on some G/H but modeled only on $\langle H \rangle_G$ according to (11) (see the survey [10] for the standard approach). For instance, we fix some entry just below the diagonal and define $B(n) \subset P \subset SL(n, \mathbb{R})$ by allowing only that entry be nonzero below the diagonal. Then P is a subgroup and $ord(P/B) = n$. We can allow more than one entry below the diagonal but the locations of these entries are not arbitrary. For $P = SL(n, \mathbb{R})$, however, $ord(P/B) = 2$ (see (26) in [2]).

Definition 13 A *projective geometry* on M is a *phg* $\varepsilon\mathcal{G}_k$ on M with vertex class $\langle B(n) \rangle_P$ for some Lie group $B(n) \subset P \subset SL(n, \mathbb{R})$ as defined above.

Observe that k is determined by $\langle B(n) \rangle_P$ and $\dim M = \dim P - \dim M$.

In fact we can choose $P = G$ any Lie group $B(n) \subset G$ not necessarily contained in $SL(n, \mathbb{R})$. It is our decision whether such a geometry (if it exists at all as we require effectiveness) will qualify as a "projective geometry". In the same way we define also a conformal geometry. As long as we recognize the stabilizer, we can define *that* geometry with the freedom of choosing the vertex class. This process is very similar to the classification of the principal bundles with some structure group H and base M and the decision of the vertex class may be interpreted as a "geometrization" condition for the total space of that principal bundle which is essentially a topological object.

We now turn back again to the imbedding (4). The filtration on the RHS of (4) in terms of the projection of jets induces a filtration inside G . Can we recover this filtration group theoretically?

For some $(G, M) \simeq G/H$ we set $H = H_0$, \mathfrak{g} = the Lie algebra of G and define inductively

$$H_{i+1} \stackrel{def}{=} \{h \in H_i \mid Ad(h)x - x \in \mathfrak{h}_i \text{ for all } x \in \mathfrak{g}, i \geq 0\} \quad (12)$$

where \mathfrak{h}_i denotes the Lie algebra of H_i . Now $H_{i+1} \triangleleft H_i$ is a normal subgroup and we obtain the filtrations

$$\begin{aligned} \dots &\subset H_1 \subset H_0 \subset G \\ \dots &\subset \mathfrak{h}_1 \subset \mathfrak{h}_0 \subset \mathfrak{g} \end{aligned} \quad (13)$$

We can define the second filtration in (13) also as

$$\mathfrak{h}_{i+1} \stackrel{def}{=} \{h \in \mathfrak{h}_i \mid [h, x] \in \mathfrak{h}_i \text{ for all } x \in \mathfrak{g}\} \quad (14)$$

If $0 \neq \mathfrak{h}_r = \mathfrak{h}_{r+1}$ for some r , then $\{1\} \neq \cap_{i \geq r} H_i \triangleleft G$ which contradicts the effectiveness of G/H . The smallest integer k such that $\mathfrak{h}_k = 0$ is called the infinitesimal order of G/H in [2]. Since $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$ for $i, j \geq 0$, $\mathfrak{h}_i/\mathfrak{h}_{i+1}$ is abelian for $i \geq 1$ and it is easy to see that $\mathfrak{h}_1 \subset Nil(\mathfrak{h})$ = the maximal nilpotent ideal of \mathfrak{h} . In particular, $k = 1$ if \mathfrak{h} is semisimple. If $H_k \neq \{1\}$ then $H_{k+1} = \{1\}$ because $H_{k+1} \subset Ker(Ad) = Z(G)$ = the center of G since G is connected which implies $H_{k+1} \subset Z(G) \cap H = \{1\}$ since G/H is effective. However, if $G/H = M$ is also simply connected then $H_k = \{1\}$. Now using (14) we define $ord(\mathfrak{g}, \mathfrak{h})$ and obtain $ord(\mathfrak{g}, \mathfrak{h}) = ord(G/H)$ if $G/H = M$ is simply connected (which we assume henceforth in this section). Now we have the commutative diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & K_{r,s}(n) & \rightarrow & G_r(n) & \xrightarrow{\pi_{i,j}} & G_s(n) \rightarrow 1 \\ & & \uparrow & & \uparrow j_r & & \uparrow j_s \\ 1 & \rightarrow & H_s/H_r & \rightarrow & H_0/H_r & \xrightarrow{\pi} & H_0/H_s \rightarrow 1 \end{array} \quad (15)$$

where $r \leq s + 1$ and the vertical imbeddings are induced by j_k . The principal bundle $G/H_i \rightarrow G/H_j$ with structure group H_j/H_i can now be identified with the principal bundle $\mathcal{G}_i^{e,\bullet} \rightarrow \mathcal{G}_j^{e,\bullet}$ as

$$\begin{array}{ccc} \mathcal{G}_i^{e,\bullet} & \rightarrow & \mathcal{G}_j^{e,\bullet} \\ \parallel & & \parallel \\ G/H_i & \rightarrow & G/H_j \end{array} \quad (16)$$

From the definition of the filtration (14) we deduce

Proposition 14 $ord(\mathfrak{g}, \mathfrak{h}_i) = k - i$, $0 \leq i \leq k$.

Proposition 14 together with (16) will play a fundamental role in Section 7.

3 The algebroid of a phg

Since $\varepsilon\mathcal{G}_k$ is a groupoid, we can define its algebroid by linearization. Our purpose is to describe this linearization process in some detail. Since $\varepsilon\mathcal{G}_k \subset \mathcal{U}_{k+1}$ is a subgroupoid, this inclusion will hold also for the algebroids. Therefore we will first recall the algebroid of \mathcal{U}_{k+1} and refer to [16], [17] for more details.

Let $T \rightarrow M$ be the tangent bundle and $J_k T \rightarrow M$ the k 'th jet extension of $T \rightarrow M$. Now $J_k T \rightarrow M$ is a vector bundle whose fiber $(J_k T)^p$ above $p \in M$ consists of the k -jets of vector fields defined near p . We have $J_0 T = T$. The sections of $J_k T \rightarrow M$ are endowed with a bracket $[\cdot, \cdot]$ called the Spencer bracket which makes $J_k T \rightarrow M$ the algebroid of \mathcal{U}_k . To define $[\cdot, \cdot]$ we need two concepts. The first is the ordinary Spencer operator

$$D : J_k T \rightarrow \wedge^* T \otimes J_{k-1} T \quad k \geq 1 \quad (17)$$

defined locally by the formula

$$(\xi^i, \xi_{j_1}^i, \dots, \xi_{j_k \dots j_1}^i) \rightarrow \left(\frac{\partial \xi^i}{\partial x^r} - \xi_r^i, \dots, \frac{\partial \xi_{j_{k-1} \dots j_1}^i}{\partial x^r} - \xi_{r j_{k-1} \dots j_1}^i \right) \quad (18)$$

The second is the algebraic bracket

$$\{ \cdot, \cdot \}_p : (J_k T)^p \times (J_k T)^p \rightarrow (J_{k-1} T)^p \quad k \geq 1 \quad (19)$$

which is defined locally by differentiating the usual bracket formula $[\xi(x), \eta(x)]^i(x) = \xi^a(x) \frac{\partial \eta^i(x)}{\partial x^a} - \eta^a(x) \frac{\partial \xi^i(x)}{\partial x^a}$ of two vector fields $\xi = (\xi^i(x))$, $\eta = (\eta^i(x))$ k -times, evaluating at $x = p$ and replacing all derivatives by jet variables. This bracket does *not* endow $J_k(T)^p$ with a Lie algebra structure as it reduces the order of jets by one. However, let $\bar{J}_k T \rightarrow M$ denote the kernel of $J_k T \rightarrow J_0 T$, that is, the fiber $(\bar{J}_k T)^p$ consists of all points in $(J_k T)^p$ which project to zero on the tangent space. Now $\{ \cdot, \cdot \}_p$ restricts to $(\bar{J}_k T)^p \times (\bar{J}_k T)^p \rightarrow (\bar{J}_k T)^p$ and $(\bar{J}_k T)^p$ endowed with $\{ \cdot, \cdot \}_p$ is a Lie algebra. In fact, this Lie algebra is the Lie algebra of the vertex group $\mathcal{U}_k^{p,p}$. Clearly $\{ \cdot, \cdot \}_p$ extends to a bracket on the sections of $J_k T \rightarrow M$ by pointwise evaluation. We denote this bracket by $\{ \cdot, \cdot \}$.

The Spencer bracket is defined now as follows. Let ξ_k, η_k be two sections of $J_k T \rightarrow T$. We lift ξ_k, η_k to some sections ξ_{k+1}, η_{k+1} of $J_{k+1} T \rightarrow M$ and define

$$[\xi_k, \eta_k] \stackrel{def}{=} \{\xi_{k+1}, \eta_{k+1}\} + i(\xi_0)D\eta_{k+1} - i(\eta_0)D\xi_{k+1} \quad k \geq 0 \quad (20)$$

where ξ_0 is the projection of ξ_k on the tangent space and $i(\xi_0)$ denotes contraction with respect to ξ_0 . The Spencer bracket $[\xi_k, \eta_k]$ does not depend on the lifts. This bracket satisfies the Jacobi identity. Further it commutes with the projections $J_k T \rightarrow J_r T$, $r \leq k$ and gives the usual bracket of vector fields for $J_0 T \rightarrow M$. It also commutes with the prolongation of vector fields: Some $\xi = (\xi^i) \in \mathfrak{X}(M)$ prolongs to a section of $J_k T \rightarrow T$ as $pr_k : (\xi^i) \rightarrow (\xi^i, \frac{\partial \xi^i}{\partial x^{j_1}}, \dots, \frac{\partial^k \xi^i}{\partial x^{j_k \dots j_1}})$ and we have $pr_k[\xi, \eta] = [pr_k \xi, pr_k \eta]$.

The above definition of the Spencer bracket is technical. To understand the geometry behind it, we first observe that the vector bundle $J_k T \rightarrow M$ is associated with the groupoid \mathcal{U}_{k+1} in the sense that any $(k+1)$ -arrow $j_{k+1}(f)^{p,q}$ induces an isomorphism

$$j_{k+1}(f)^{p,q} : (J_k T)^p \rightarrow (J_k T)^q \quad (21)$$

defined locally by differentiating the transformation rule $\frac{\partial y^i}{\partial x^a} \eta^a(x) = \eta^i(y)$ of the vector field $\eta = (\eta^i)$ k -times, evaluating at $x = p$, $y = q$, and replacing all derivatives with jet variables. In particular (21) gives a faithful representation of $\mathcal{U}_{k+1}^{p,p} \simeq G_{k+1}(n)$ on $(J_k T)^p$. This representation descends to a representation of $\mathcal{U}_k^{p,p} \simeq G_k(n)$ on $(J_k T)^p \simeq \mathfrak{g}_k(n)$ which is the adjoint representation of $G_k(n)$ on its Lie algebra $\mathfrak{g}_k(n)$.

Now we recall that the points of the principal bundle $\pi : \mathcal{U}_k^{e,\bullet} \rightarrow M$ are k -arrows emanating from the base point e . So let $\bar{p} \in \mathcal{U}_k^{e,\bullet}$, $\pi(\bar{p}) = p = j_k(f)^{e,p}$. Let $\xi_{\bar{p}}$ be a tangent vector at \bar{p} which projects to the tangent vector ξ_p at p . By acting with the structure group $\mathcal{U}_k^{e,e}$ on the fiber $\mathcal{U}_k^{e,p}$ we translate $\xi_{\bar{p}}$ to the points in the fiber $\pi^{-1}(p) = \mathcal{U}_k^{e,p}$. We call this data a set of parallel vectors at the fiber $\pi^{-1}(p)$. We now have

Proposition 15 *There is a canonical identification between the following objects.*

- i) *The set of parallel vectors at the fiber $\pi^{-1}(p)$*
- ii) *The fiber $(J_k T)^p$ of the vector bundle $J_k T \rightarrow M$ over p .*

In particular, Proposition 15 gives the canonical identification

$$T(\mathcal{U}_k^{e,\bullet})_{\bar{p}} \simeq (J_k T)^p \quad (22)$$

where $T(\mathcal{U}_k^{e,\bullet})_{\bar{p}}$ denotes the tangent space of $\mathcal{U}_k^{e,\bullet}$ at \bar{p} .

The reason for (22) is simple: Let $\bar{p} \in \mathcal{U}_k^{e,\bullet}$, $\pi(\bar{p}) = p = j_k(f)^{e,p}$. A diffeomorphism g on M lifts to a diffeomorphism \bar{g} on $\mathcal{U}_k^{e,\bullet}$ defined by $j_k(f)^{e,p} \rightarrow j_k(g \circ f)^{e,g(p)}$. Consider the 1-parameter group $g_t(x)$ of local diffeomorphisms defined by a vector field $\xi(x)$ defined around p . Now $\xi(x)$ lifts to a vector field on $\mathcal{U}_k^{e,\bullet}$ whose value at \bar{p} depends on $j_k(\xi)^p$ and this map is an isomorphism.

Proposition 15 gives the canonical identification

$$\{\text{right invariant vector fields on } \mathcal{U}_k^{e,\bullet} \rightarrow M\} \simeq \{\text{sections of } J_k T \rightarrow M\} \quad (23)$$

Since the LHS of (23) is a Lie algebra with the usual bracket of vector fields on $\mathcal{U}_k^{e,\bullet}$, we get a bracket on the RHS of (23)...which is the Spencer bracket.

The linearization of the nonlinear PDE $\varepsilon\mathcal{G}_k$ to the linear PDE $\varepsilon\mathfrak{G}_k$ is best understood in coordinates. We replace the "finite transformations" in (2) by "infinitesimal transformations", i.e., we substitute $y^i = x^i + t\xi^i$, $f_{j_1}^i = \delta_{j_1}^i + t\xi_{j_1}^i$, $f_{j_2 j_1}^i = t\xi_{j_2 j_1}^i$, ..., $f_{j_{k+1} \dots j_1}^i = t\xi_{j_{k+1} \dots j_1}^i$ into (2) and differentiate at $t = 0$. The resulting equations

$$\widehat{\Phi}^\alpha : (x^i, \xi^i, \dots, \xi_{j_{k+1} \dots j_1}^i) = 0 \quad (24)$$

are linear in the variables $\xi^i, \dots, \xi_{j_{k+1} \dots j_1}^i$ which are the local coordinates on $J_{k+1}T$ over the fiber $\pi^{-1}(x)$. As in (2), the top coordinates $\xi_{j_{k+1} \dots j_1}^i$ can be solved uniquely in terms of $\xi^i, \dots, \xi_{j_k \dots j_1}^i$ (we will denote this splitting again by ε) and (24) puts no restriction on the variables x^i . So (24) defines a subbundle $\varepsilon\mathfrak{G}_k \rightarrow M$ of $J_{k+1}T \rightarrow M$. The crucial fact is that the Spencer bracket restricts to the sections of $\varepsilon\mathfrak{G}_k \rightarrow M$. Since $\mathcal{G}_k \rightarrow \varepsilon\mathcal{G}_k$ is an isomorphism of groupoids, $\mathfrak{G}_k \rightarrow \varepsilon\mathfrak{G}_k$ is an isomorphism of algebroids, i.e., it preserves the bracket. Thus we have the diagram

$$\begin{array}{ccc} \mathcal{U}_{k+1} & \Longrightarrow & J_{k+1}T \rightarrow M \\ \cup & & \cup \\ \varepsilon\mathcal{G}_k & \Longrightarrow & \varepsilon\mathfrak{G}_k \rightarrow M \end{array} \quad (25)$$

where \Longrightarrow denotes linearization.

At this point, it is possible to define the Lie algebroid $\varepsilon\mathfrak{G}_k \rightarrow M$ independently as an "infinitesimal phg of order k on M ". However, once properly defined, this object will be the linearization of a unique "finite phg $\varepsilon\mathcal{G}_k$ ".

Like the Spencer bracket, all the calculus on $J_{k+1}T \rightarrow M$ (which we only touched here) restricts to $\varepsilon\mathfrak{G}_k \rightarrow M$. For instance, for $j_{k+1}(f)^{p,q} \in \varepsilon\mathcal{G}_k^{p,q}$, (21) restricts as

$$\varepsilon j_{k+1}(f)^{p,q}_* : \mathfrak{G}_k^p \rightarrow \mathfrak{G}_k^q \quad (26)$$

where \mathfrak{G}_k^p denotes the fiber of $\mathfrak{G}_k \rightarrow M$ over p . Therefore, even though $J_k T \rightarrow M$ is associated with \mathcal{U}_{k+1} , $\mathfrak{G}_k \rightarrow M$ is associated with \mathcal{G}_k . This is a particular case of the "stabilization of the order of jets using the splitting ε " which will play a fundamental role in this note. Similarly, the identification (22) restricts as

$$T(\mathcal{G}_k^{e,\bullet})^{\overline{p}} \simeq \mathfrak{G}_k^p \quad (27)$$

Now (24) shows that the points \overline{p} in the fiber \mathfrak{G}_k^p of $\varepsilon\mathfrak{G}_k \rightarrow M$ over p are "initial conditions" for the linear PDE $\varepsilon\mathfrak{G}_k \rightarrow M$ locally defined by (24). We call $\varepsilon\mathfrak{G}_k \rightarrow M$ locally solvable at $p \in M$ if for any $\overline{p} \in \mathfrak{G}_k^p$ there exists a vector field ξ defined near p whose prolongation $pr_{k+1}(\xi)$ is a section of $\varepsilon\mathfrak{G}_k \rightarrow M$ passing through \overline{p} .

Definition 16 $\varepsilon\mathfrak{G}_k \rightarrow M$ is locally solvable if it is locally solvable on M .

If $\varepsilon\mathfrak{G}_k \rightarrow M$ is locally solvable, then its solutions are determined locally by their initial conditions. Therefore "analytic continuation" is possible along paths. Note that the Spencer bracket becomes the ordinary bracket of vector fields on local solutions. Thus we can define the presheaf of Lie algebras $\mathfrak{g}(U) \stackrel{def}{=} \text{the Lie algebra of local solutions on } U$.

Now we have the following fundamental

Proposition 17 $\varepsilon\mathcal{G}_k \rightarrow M$ is locally solvable $\iff \varepsilon\mathfrak{G}_k \rightarrow M$ is locally solvable.

The implication \Rightarrow follows easily from definitions whereas \Leftarrow is quite non-trivial. To see what is involved in Proposition 17, assume that $\varepsilon\mathcal{G}_k \rightarrow M$ is locally solvable and the pseudogroup $\varepsilon\mathcal{G}_k$ globalizes to G so that $\mathcal{G}_k^{e,\bullet} \simeq G$. This implies that the local solutions $\widetilde{\varepsilon\mathfrak{G}_k}$ of $\varepsilon\mathfrak{G}_k \rightarrow M$ also globalize and we obtain a Lie algebra \mathfrak{g} of vector fields on M . Not surprisingly, \mathfrak{g} is the Lie algebra of the infinitesimal generators of the Klein geometry (G, M) . Since (G, M) is effective by construction, \mathfrak{g} is isomorphic to the Lie algebra of the abstract Lie group G .

So the bottom line of (25) becomes the assignment

$$\{\text{The Klein geometry } (G, M)\} \Rightarrow \{\text{the Lie algebra } \mathfrak{g} \text{ of the infinitesimal generators}\} \quad (28)$$

So \Leftarrow of Proposition 17 asserts that locally (G, M) can be recovered from \mathfrak{g} . For an abstract Lie group G , observe that the assignment $\{G\} \Rightarrow \{\text{its Lie algebra } \mathfrak{g}\}$ involves a choice of left/right and is not canonical whereas (28) is canonical even for $k = 0$, i.e., if G acts simply transitively. In this simplest case, \Leftarrow of Proposition 17 becomes the classical version (not the Cartan's version) of the Lie's 3rd Theorem.

Finally we note that, assuming local solvability, **A1** linearizes to an injection of Lie algebras and we get a correspondence between the algebraic filtration in (14) and the "jet filtration" on $\mathfrak{G}_k \rightarrow M$ in terms of the projection of jets. In particular we get the linearizations of (15), (16).

4 Curvature

Let $\varepsilon\mathcal{G}_k$ be a phg of order k on M . In Sections 2, 3 we have seen that the local solvability of $\varepsilon\mathcal{G}_k$ and its algebroid $\varepsilon\mathfrak{G}_k$ is a fundamental concept because the theory of locally solvable phg 's is the same as the theory of homogeneous spaces. Since local solvability is a very intuitive concept, we can easily guess at some constructions and theorems assuming local solvability without going into technical proofs. Since this is a qualitative concept, it is desirable to define a quantity $\mathcal{R}_k = \text{the curvature of } \varepsilon\mathcal{G}_k$ in such a way that we will have

$$\mathcal{R}_k = 0 \iff \varepsilon\mathcal{G}_k \text{ is locally solvable} \quad (29)$$

Similarly we want to define \mathfrak{R}_k = the curvature of $\varepsilon\mathfrak{G}_k$ such that

$$\mathfrak{R}_k = 0 \iff \varepsilon\mathfrak{G}_k \text{ is locally solvable} \quad (30)$$

In view of Proposition 17 we will have

$$\mathcal{R}_k = 0 \iff \mathfrak{R}_k = 0 \quad (31)$$

Further, since $\varepsilon\mathfrak{G}_k$ is the linearization of $\varepsilon\mathcal{G}_k$, we require that \mathfrak{R}_k should be obtained from \mathcal{R}_k by the same linearization process.

Probably the first thing that comes to mind is the following: Consider the first prolongation $J_1\mathcal{G}_k^{e,\bullet} \rightarrow \mathcal{G}_k^{e,\bullet}$ of the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$. Sections of $J_1\mathcal{G}_k^{e,\bullet} \rightarrow \mathcal{G}_k^{e,\bullet}$ are in 1-1 correspondence with connections on $\mathcal{G}_k^{e,\bullet} \rightarrow M$. Now it is easy to show

Proposition 18 *The splitting $\mathcal{G}_k \rightarrow \varepsilon\mathcal{G}_k$ defines a connection on $\mathcal{G}_k^{e,\bullet} \rightarrow M$.*

It is natural to define \mathcal{R}_k to be the curvature of this connection. It is an extremely surprising fact (probably more than that!) that \mathcal{R}_k is not this curvature!! There is a very short conceptual way of seeing this as follows: For simplicity we assume $\mathcal{R}_k=0$ and $\widetilde{\varepsilon\mathcal{G}_k}$ globalizes to G so that $\mathcal{G}_k^{e,\bullet} \simeq G$ and G acts transitively on M . However in the general theory of principal bundles, the total space of the principal bundle, in particular $\mathcal{G}_k^{e,\bullet} \simeq G$ in our case, does not act on the base manifold...so these two curvatures can not be the same objects in general. This will follow also from the technical definition (35) of \mathcal{R}_k below. At this point it is crucial to observe that the principal bundle and the connection are separate objects in the general theory whereas they unify into a single object in the definition of a *phg*. In particular, \mathcal{R}_k is not the curvature of a connection on any principal bundle but is the curvature of $\varepsilon\mathcal{G}_k$.

To find the technical definition of \mathcal{R}_k , we consider the first nonlinear Spencer sequence ([12], [16], [17])

$$1 \longrightarrow \text{Aut}(M) \xrightarrow{j_k} \mathcal{U}_k \xrightarrow{D_1} T^* \otimes J_{k-1}T \xrightarrow{D_2} \wedge^2 T^* \otimes J_{k-2}T \quad (32)$$

The explicit local formulas describing D_1 , D_2 are given in [17], pg.213-216 and it is quite easy to do computations with these formulas (if we have enough patience!). Note that $k \geq 2$ in (32). Observe that D_1 , D_2 reduce the order of jets by one.

Now (32) restricts as

$$\mathcal{G}_k \xrightarrow{D_1} T^* \otimes \mathfrak{G}_{k-1} \xrightarrow{D_2} \wedge^2 T^* \otimes \mathfrak{G}_{k-2} \quad (33)$$

and still $k \geq 2$ in (33). The crucial fact is that the splittings $\varepsilon : \mathcal{G}_k \rightarrow \varepsilon\mathcal{G}_k$ and $\varepsilon : \mathfrak{G}_k \rightarrow \varepsilon\mathfrak{G}_k$ stabilize the order of jets in (33) as

$$\mathcal{G}_k \xrightarrow{D'_1} T^* \otimes \mathfrak{G}_k \xrightarrow{D'_2} \wedge^2 T^* \otimes \mathfrak{G}_k \quad (34)$$

and now $k \geq 0$ in (34). Even though $D_2 \circ D_1 = 0$ in (33), we do not have $D'_2 \circ D'_1 = 0$ in (34). We define

$$\mathcal{R}_k \stackrel{def}{=} D'_2 \circ D'_1 \quad (35)$$

So for any k -arrow $\alpha_k^{p,q} \in \mathcal{G}_k^{p,q}$ and $\xi_p, \eta_p \in T_p(M)$, we have

$$\mathcal{R}_k(\alpha_k^{p,q})(\xi_p, \eta_p) \in \mathfrak{G}_k^q \quad (36)$$

where, as before, \mathfrak{G}_k^q denotes the fiber of the algebroid $\mathfrak{G}_k \rightarrow M$ over $q \in M$.

We now have

Proposition 19 *The following are equivalent*

- i) $\mathcal{R}_k=0$
- ii) $\varepsilon\mathcal{G}_k$ is locally solvable
- iii) (34) is locally exact at $T^* \otimes \mathfrak{G}_k$
- iv) $\varepsilon\mathcal{G}_k$ is involutive

If one of the conditions of Proposition 19 holds, then (34) extends to the second nonlinear Spencer sequence

$$\mathbf{1} \longrightarrow \widetilde{\varepsilon\mathcal{G}_k} \xrightarrow{j_k} \mathcal{G}_k \xrightarrow{D'_1} T^* \otimes \mathfrak{G}_k \xrightarrow{D'_2} \wedge^2 T^* \otimes \mathfrak{G}_k \quad (37)$$

which is locally exact. It is instructive to check (37) in the simplest case $k=0$ of parallelizable manifolds studied in detail in [1] and construct (37) in this case. Observe that there is no curvature in (37) (see however [17], pg. 216 where the operator D'_2 is claimed to be the curvature).

Now we come to \mathfrak{R}_k . Let $\bar{p} = \alpha_k^{e,p}$ and $\bar{q} = \beta_k^{e,q}$ be two points of $\mathcal{G}_k^{e,\bullet}$. According to (36) we have

$$\mathcal{R}_k(\bar{q} \circ \bar{p}^{-1})(\xi_p, \eta_p) \in \mathfrak{G}_k^q \quad (38)$$

We now fix \bar{p}, ξ_p, η_p in (38) and let \bar{q} approach \bar{p} along the direction of some tangent vector $\sigma_k^p \in \mathfrak{G}_k^p = T(\mathcal{G}_k^{e,\bullet})^{\bar{p}}$. The limiting value is an element of \mathfrak{G}_k^p which we write as

$$(\mathfrak{R}_k(\bar{p})(\xi_p, \eta_p))(\sigma_k^p) \quad (39)$$

The function $\sigma_k^p \rightarrow (\mathfrak{R}_k(\bar{p})(\xi_p, \eta_p))(\sigma_k^p)$ turns out to be linear and therefore

$$\mathfrak{R}_k(\bar{p})(\xi_p, \eta_p) \in \text{Hom}(\mathfrak{G}_k^p, \mathfrak{G}_k^p) \quad (40)$$

So the object \mathfrak{R}_k assigns to a point \bar{p} on the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ and two tangent vectors ξ_p, η_p at $p \in M$ a linear map on the tangent space $T(\mathcal{G}_k^{e,\bullet})^{\bar{p}}$. Observe that \mathfrak{R}_k is not an ordinary 2-form on M .

Since (40) is the linearization of (36) which arises from the second nonlinear Spencer sequence, we should be able to derive \mathfrak{R}_k directly from the second linear Spencer sequence. Indeed, the ordinary Spencer operator (17) restricts to

$$\mathfrak{G}_k \xrightarrow{D} T^* \otimes \mathfrak{G}_{k-1} \quad (41)$$

where $k \geq 1$. The splitting $\varepsilon : \mathfrak{G}_k \rightarrow \varepsilon \mathfrak{G}_k$ stabilizes the order of jets in (41) as

$$\mathfrak{G}_k \xrightarrow{D'} T^* \otimes \mathfrak{G}_k \quad (42)$$

Acting with d on T^* and with D' on \mathfrak{G}_k we extend (42) one step the right as

$$\mathfrak{G}_k \xrightarrow{D'} T^* \otimes \mathfrak{G}_k \xrightarrow{D''} \wedge^2 T^* \otimes \mathfrak{G}_k \quad (43)$$

where $D'' = D'_2$ in (37). Clearly we have

$$\mathfrak{R}_k = D'' \circ D' \quad (44)$$

Now (43) extends to the sequence

$$\mathfrak{G}_k \xrightarrow{D'} T^* \otimes \mathfrak{G}_k \xrightarrow{D''} \wedge^2 T^* \otimes \mathfrak{G}_k \longrightarrow \dots \longrightarrow \wedge^n T^* \otimes \mathfrak{G}_k \quad (45)$$

which is locally exact if $\mathfrak{R}_k = 0$ and (45) is the linear second Spencer sequence. To summarize, \mathcal{R}_k and \mathfrak{R}_k are obstructions to the passage from the first to the second Spencer sequences.

Finally, let $P = \mathcal{G}_k^{e,\bullet}$, $H = \mathcal{G}_k^{e,e}$ and \mathfrak{h} the Lie algebra of H . A connection on $P \rightarrow M$ is an \mathfrak{h} -valued 1-form on P and its curvature R is an \mathfrak{h} -valued 2-form on P . Assume that $\mathfrak{R}_k = 0$ and $\mathcal{G}_k^{e,\bullet}$ globalizes to G which acts transitively on M . Let \mathfrak{g} be the Lie algebra of G . Now (40) shows that \mathfrak{R}_k is a 2-form on P with values in $\text{Hom}(\mathfrak{g}, \mathfrak{g})$ and therefore $\mathfrak{R}_k \neq R$ in general (however, see Section 10 for the remarkable case $k = 0$!). Note that \mathfrak{R}_k can not be also the curvature of a Cartan connection on $\mathcal{G}_k^{e,\bullet} \rightarrow M$ which is a \mathfrak{g} -valued 1-form on $\mathcal{G}_k^{e,\bullet}$ and its curvature is a \mathfrak{g} -valued 2-form. However, we will see in Section 9 that *this will be the case* with a rather restrictive assumption.

5 Cartan algebroids

In [3], [4], Blaom proposed a very interesting and general theory of infinitesimal geometric structures. The general philosophy, which he attributes to E. Cartan, is to view an infinitesimal geometric structure as a symmetry deformed by curvature. For this purpose, he defines the concept of the Cartan algebroid. These are algebroids equipped with a linear connection (which he calls somewhat confusingly the Cartan connection) whose covariant derivative is compatible with the algebroid structure. The curvature of the Cartan connection vanishes if and only if M is locally homogeneous. As an important fact, a Cartan algebroid is defined without the use of jets and need not be transitive!

We now have

Proposition 20 *The algebroid $\varepsilon \mathfrak{G}_k \rightarrow M$ of the phg $\varepsilon \mathcal{G}_k$ is a transitive Cartan algebroid. The first operator D' in (45) is the Cartan connection of $\varepsilon \mathfrak{G}_k \rightarrow M$ and \mathfrak{R}_k defined by (44) is its curvature.*

We believe that all transitive Cartan algebroids arise as the Lie algebroids of phg 's (possibly with some mild conditions of regularity). Therefore, we believe that the theory of Cartan groupoids, whose study is initiated in the Appendix A of [4] and expanded in [6], is essentially the same as the theory of phg 's *in the transitive case*.

In the recent preprint [5], Blaom also clarifies the the concept of completeness of a not necessarily flat Cartan algebroid. We believe that this will have important consequences for the theory of PDE 's in view of the "equivalence" of transitive Cartan algebroids and phg 's.

6 Characteristic classes on the base

One of the great achievements of global the differential geometry in this century is the theory of characteristic classes on principal and vector bundles. These classes are cohomology classes on the base manifold which measure the twisting of the bundle, i.e., their deviation from being globally trivial. This is a topological theory. Therefore it came as a great surprise when in 1970 R. Bott showed that Chern classes are also obstructions to integrability of the to plane fields, i.e., subbundles of the tangent bundle.

Now let $\varepsilon\mathcal{G}_k$ be a phg of order k and consider the algebroid $\mathfrak{G}_k \rightarrow M$. Let $\mathcal{P}^*(M, \mathfrak{G}_k)$ denote the Pontryagin algebra (P -algebra for short) of the vector bundle $\mathfrak{G}_k \rightarrow M$. To recall the definition of $\mathcal{P}^*(M, \mathfrak{G}_k)$, let m denote the fiber dimension of $\mathfrak{G}_k \rightarrow M$. Let $\mathfrak{gl}(m, \mathbb{R})$ denote the Lie algebra of $GL(m, \mathbb{R}) = G_1(m)$ = the Lie group of invertible $m \times m$ matrices. So $\mathfrak{gl}(m, \mathbb{R})$ is the linear Lie algebra of all $m \times m$ matrices. A polynomial function $\varphi : \mathfrak{gl}(m, \mathbb{R}) \rightarrow \mathbb{R}$ is called invariant if $\varphi(gAg^{-1}) = \varphi(A)$ for all $A \in \mathfrak{gl}(m, \mathbb{R})$, $g \in GL(m, \mathbb{R})$. The vector space $I_{GL(m, \mathbb{R})}$ of all invariant polynomials is an algebra generated by $T_j(A) \stackrel{def}{=} \text{Trace}(A^j)$, $j \geq 0$. Now any connection ω on the vector bundle $\mathfrak{G}_k \rightarrow M$ defines the algebra homomorphism CW

$$\begin{aligned} CW &: I_{GL(m, \mathbb{R})} \longrightarrow H_{dR}^*(M, \mathbb{R}) \\ &: \varphi \longrightarrow \varphi(\kappa) \end{aligned} \tag{46}$$

where κ is the curvature of ω . The map (46) is independent of the connection. If φ is homogeneous of degree r , then $\varphi(\kappa) \in H_{dR}^{2r}(M, \mathbb{R})$. Now $\mathcal{P}^*(M, \mathfrak{G}_k)$ is the image of CW and it can be shown (see [7]) that $\mathcal{P}^j(M, \mathfrak{G}_k) = 0$ if j is not divisible by 4.

As we observed in Section 3, the splitting $\mathfrak{G}_k \rightarrow \varepsilon\mathfrak{G}_k$ defines a the particular linear connection ε on the vector bundle $\mathfrak{G}_k \rightarrow M$ with curvature \mathfrak{R}_k . Further, $\mathfrak{R}_k = 0 \Leftrightarrow \mathcal{R}_k = 0 \Leftrightarrow \varepsilon\mathcal{G}_k$ is locally solvable by Propositions 17,19. Since the map (46) is independent of the connection, we obtain

Proposition 21 *If $\varepsilon\mathcal{G}_k$ is locally solvable, then $\mathcal{P}^*(M, \mathfrak{G}_k) = 0$.*

If $k = 0$, note that $\mathcal{P}^*(M, T) = 0$ without the assumption of local solvability of $\varepsilon\mathcal{G}_0$ since the existence of $\varepsilon\mathcal{G}_0$ is equivalent to the parallelizability of M .

Proposition 21 follows almost trivially from our definitions. However it gives a totally new way of looking at characteristic classes: among the set of all vector bundles over M , there is a particular subset of vector bundles with the property that the restriction of the functor \mathcal{P}^* to this subset gives global obstructions to integrability in the sense of local solvability.

Recalling that $\mathfrak{G}_0 = T$, we have the exact sequence of vector bundles

$$0 \longrightarrow \mathcal{I}_k \longrightarrow \mathfrak{G}_k \longrightarrow T \longrightarrow 0 \quad (47)$$

For $k = 0$, $\mathfrak{G}_0 = T$ and $\mathcal{I}_0 = 0$. For $k \geq 1$, $\mathcal{I}_k \rightarrow M$ is a bundle of Lie algebras whose fiber over p consists of all k -jets of vector fields at p which project to zero on the tangent space at p and this fiber is the Lie algebra of the vertex group $\mathcal{G}_k^{p,p}$. From (47) we conclude $\mathfrak{G}_k = \mathcal{I}_k \oplus T$. Therefore if $\varepsilon\mathcal{G}_k$ is locally solvable, then $p(\mathcal{I}_k) \cdot p(T) = 1$ which is the first indication that the existence of some locally solvable $\varepsilon\mathcal{G}_k$ puts restrictions on $\mathcal{P}^*(M, T)$. To dig this point deeper, we will recall some facts from [9] which have an intriguing relation to Proposition 21.

Let $J_{(m)}T \rightarrow M$ denote the m -th order iterated jet bundle of $T \rightarrow M$, i.e., $J_{(m)}T \rightarrow M$ is obtained from $T \rightarrow M$ by applying the functor J_1 successively m -times. So $J_{(0)}T = T$, $J_{(1)}T = J_1T$ but $J_mT \rightarrow M$ is a subbundle of $J_{(m)}T \rightarrow M$ for $m \geq 2$. If $J_{(m)}T \rightarrow M$ admits a flat linear connection for some m , then so does $J_{(k)}T \rightarrow M$ for $k \geq m$. The smallest such integer $\alpha(M)$, if it exists, is called the Andreotti invariant of M . If M is m -flat for some M , then $\mathcal{P}^*(M, T) = 0$. Therefore $\mathcal{P}^*(M, T) \neq 0 \Rightarrow \alpha(M) = \infty$, like projective spaces. Now [9] shows that $\alpha(M)$ is finite for certain lens spaces and makes a detailed study in this case.

The reason why m -flatness forces $\mathcal{P}^*(M, T) = 0$ as stated in [9] can be shown as follows. The structure group $j_{(m)}G_1(n)$ of $J_{(m)}T \rightarrow M$ can be reduced to $G_1(n)$ because $j_{(m)}G_1(n) = G_1(n) \rtimes E$ for some subgroup E diffeomorphic to some \mathbb{R}^d . This reduction $F \rightarrow M$ is isomorphic to the direct sums of certain tensor products of T and T^* . For $m = 2$, for instance, $F = T \oplus (T \otimes T^*) \oplus (T \otimes T^* \otimes T^*)$ which is easily checked by the chain rule. Since $J_{(m)}T \rightarrow M$ and $F \rightarrow M$ are isomorphic we have $\mathcal{P}^*(M, J_{(m)}T) = \mathcal{P}^*(M, F)$. Observe that even though $J_{(m)}T \rightarrow M$ is a natural bundle of order $m + 1$, $F \rightarrow M$ is a tensor bundle and therefore \mathcal{P}^* is sensitive only to first order jets, a fact which will be of great importance in Section 7. Therefore, if M is m -flat, then $\mathcal{P}^*(M, F) = 0$. However the P -classes of direct sums and tensor products are determined by the P -classes of the factors separately. It follows that the P -classes of $F \rightarrow M$, which all vanish, can be expressed in terms of the P -classes of $T \rightarrow M$ which gives polynomial relations of the form

$$\begin{aligned}
ap_1(T) &= 0 \\
bp_1(T)^2 + cp_2(T) &= 0 \\
dp_1(T)^3 + ep_2(T)^2 + fp_3(T) &= 0 \\
..... &= 0
\end{aligned} \tag{48}$$

for some constants a, b, \dots . It remains to show $a \neq 0$, $c \neq 0$, $f \neq 0$ and this is done by explicit algebraic computation. This argument shows that m -flatness for some m implies $\mathcal{P}^*(M, J_{(k)}T) = 0$ for all $k \geq 0$.

Now [9] shows that $\alpha(M)$ can be arbitrarily large. The following question will be our driving force in Section 7.

Q: Suppose $\alpha(M)$ is finite so that $\mathcal{P}^*(M, J_{(k)}T) = 0$ for all $k \geq 0$. What are the obstructions to $(\alpha(M) - 1)$ -flatness?

Now consider some $\varepsilon\mathcal{G}_k$ and the principal $\varepsilon\mathcal{G}_k^{\bullet} \rightarrow M$ with $\mathcal{G}_k^{e,e}$ as the structure group which we assume to be connected. By the Iwasawa-Malcev theorem, we decompose $\mathcal{G}_k^{e,e} = KE$ where $K \subset \mathcal{G}_k^{e,e}$ is a maximal compact subgroup and the subset E is euclidean.

Lemma 22 *The restriction of the projection $\pi_{k,1} : \mathcal{G}_k^{e,e} \rightarrow \mathcal{G}_1^{e,e}$ to K is an imbedding.*

The reason is that the kernel of $\pi_{k,k-1} : \mathcal{G}_k^{e,e} \rightarrow \mathcal{G}_{k-1}^{e,e}$ is a subgroup of the vector group $K_{k,k-1}$ in (15) and K must intersect this kernel trivially since it is compact. Iterating this argument, we see that K must be contained in $\mathcal{G}_1^{e,e}$.

Let $\varepsilon\mathfrak{G}_k \rightarrow M$ be the algebroid of $\varepsilon\mathcal{G}_k$ and let $P \rightarrow M$ be the principal bundle associated with $\varepsilon\mathfrak{G}_k \rightarrow M$ with $GL(m, \mathbb{R})$ as the structure group where m is the dimension of the fibers of $\varepsilon\mathfrak{G}_k \rightarrow M$. Now $\varepsilon\mathcal{G}_k^{\bullet} \rightarrow M$ is a reduction of $P \rightarrow M$ with $\mathcal{G}_1^{e,e} \subset GL(m, \mathbb{R})$. Using Lemma 22, we can reduce the structure group further to $K \subset \mathcal{G}_1^{e,e}$. Now we have

$$\begin{array}{ccc}
I_{GL(m, \mathbb{R})} & \longrightarrow & H_{dR}^*(M, \mathbb{R}) \\
\downarrow \theta & & \parallel \\
I_K & \longrightarrow & H_{dR}^*(M, \mathbb{R})
\end{array} \tag{49}$$

where θ is the restriction homomorphism induced by the restrictions $K \subset \mathcal{G}_k^{e,e} \subset GL(m, \mathbb{R})$. Now if $\varepsilon\mathcal{G}_k$ is locally solvable, then the image of the bottom homomorphism in (49) vanishes and as above, we believe that this brings polynomial relations like (48) which depend on $\varepsilon\mathcal{G}_k$. We believe that the clarification of this scenario will explain many known vanishing phenomena, like Bott vanishing theorem for plane fields, Chern vanishing theorem which states $\mathcal{P}^*(M, T) = 0$ for a Riemannian structure of constant curvature, Borel-Hirzebruch vanishing theorem which states $\mathcal{P}^*(M, T) = 0$ for G/T where G is compact and T is a maximal torus, the well known relations between the Chern classes of projective spaces...and many other phenomena about the structure of the characteristic classes of homogeneous spaces.

7 Higher order characteristic classes

This section is the main core of this note. We will outline here the construction of higher order obstructions to local solvability. In particular, our method will give obstructions to m -flatness. We do not know whether these invariants can be nontrivial. What we do know, however, is that they will be highly nontrivial if they are nontrivial at all!

As we observed in Section 6, the P -algebra algebra $\mathcal{P}^*(M, \mathfrak{G}_k) \subset H_{dR}^*(M, \mathbb{R})$ is sensitive only to first order jets and is "topological" even though k is large. This topology persists even if $\varepsilon\mathcal{G}_k$ is locally solvable in the following sense.

Let a connected G act transitively on M . If M is compact and the stabilizers of the action are connected (this is so if M is also simply connected), then a maximal compact subgroup $K \subset G$ acts also transitively on M according to [13]. Hence $M = G/H = K/K \cap H$ for some stabilizer H . Therefore, as long as we are interested in the topological properties of a compact Klein geometry G/H with connected H , we may assume that G (and therefore H) is compact. Obviously $\text{ord}(G/H) = 0$ if $H = \{1\}$. Now Lemma 22 implies

Proposition 23 *If H is compact, connected and nontrivial, then $\text{ord}(G/H) = 1$.*

The above arguments make it clear that the cohomological invariants of $\varepsilon\mathcal{G}_k$ which depend on k can not be topological if k is large. Further, we should not search for such invariants in the cohomology of the base M . In particular, we should not consider $\varepsilon\mathcal{G}_k$ as a fibering over M as we did so far, i.e., we should not let $\varepsilon\mathcal{G}_k$ act on M .

Inspired by Proposition 14, we start with the following

Proposition 24 *Let $\varepsilon\mathcal{G}_k$ be a phg of order k on M . Then the total space $\mathcal{G}_k^{e,\bullet}$ of the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ is parallelized by ε in a canonical way.*

Proposition 24 is trivial for $k = 0$ since $\mathcal{G}_0^{e,\bullet} \simeq M$ is parallelizable by definition. If $\mathcal{R}_k = 0$ and $\widehat{\varepsilon\mathcal{G}_k}$ is globalizable, then $\mathcal{G}_k^{e,\bullet} \simeq G$ which is surely parallelizable. The surprising fact is that the statement holds for all $k \geq 0$ without the assumption $\mathcal{R}_k = 0$.

The proof follows almost trivially from the definition of a phg. First we recall the following trivial

Fact: Let M be a smooth manifold and $p, q \in M$. There is a canonical identification between the following sets:

- i) The set of 1-arrows from p to q
- ii) The set of isomorphisms $T_p(M) \rightarrow T_q(M)$

Now let \bar{p}, \bar{q} be arbitrary points on $\mathcal{G}_k^{e,\bullet}$ which project to $p, q \in M$. So \bar{p}, \bar{q} are two k -arrows from e to p, q respectively, say $\bar{p} = j_k(f)^{e,p}$ and $\bar{q} = j_k(g)^{e,q}$. Therefore $j_k(g)^{e,q} \circ [j_k(f)^{e,p}]^{-1} = j_k(g \circ f^{-1})^{p,q}$ is a k -arrow from p to q . The splitting ε gives the *unique* $(k+1)$ -arrow $\varepsilon j_k(g \circ f^{-1})^{p,q}$ from p to q . Now let $\xi_{\bar{p}}$ be a tangent vector at \bar{p} . According to (22) the tangent space $T(\mathcal{G}_k^{e,\bullet})_{\bar{p}}$ is the

same as the fiber \mathfrak{G}_k^p of the algebroid $\mathfrak{G}_k \rightarrow M$ over p . Therefore the isomorphism $\varepsilon j_k (g \circ f^{-1})^{p,q} : \mathfrak{G}_k^p \rightarrow \mathfrak{G}_k^q$ in (26) is an isomorphism $\varepsilon j_k (g \circ f^{-1})^{p,q} : T(\mathcal{G}_k^{e,\bullet})^{\bar{p}} \rightarrow T(\mathcal{G}_k^{e,\bullet})^{\bar{q}}$. The above fact implies that the object $\varepsilon j_k (g \circ f^{-1})^{p,q}$, which is a $(k+1)$ -arrow from p to q is at the same time a 1-arrow from \bar{p} to \bar{q} . In short, $(k+1)$ -arrows on M define 1-arrows on $\mathcal{G}_k^{e,\bullet}$!! So for any two points $\bar{p}, \bar{q} \in \mathcal{G}_k^{e,\bullet}$ we established a unique 1-arrow from \bar{p} to \bar{q} which is equivalent to the parallelizability of $\mathcal{G}_k^{e,\bullet}$ since this assignment is smooth and is a homomorphism of groupoids. We will continue to denote this splitting by ε .

The above proof warns us that we should be more careful with our notation. So we denote a phg $\varepsilon \mathcal{G}_k$ on M by $(\mathcal{G}_0^{e,\bullet}, \varepsilon \mathcal{G}_k) = (M, \varepsilon \mathcal{G}_k)$ henceforth. Accordingly we will denote the phg of order zero in Proposition 24 by $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathcal{G}_k)$. Let \mathcal{R}_0 denote the curvature of $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathcal{G}_k)$. By Propositions 17, 19, $\mathcal{R}_0 = 0 \Leftrightarrow (\mathcal{G}_k^{e,\bullet}, \varepsilon \mathcal{G}_k)$ integrates to the pseudogroup $(\widetilde{\mathcal{G}_k^{e,\bullet}}, \varepsilon \mathcal{G}_k)$, i.e., all 1-arrows of $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathcal{G}_k)$ integrate uniquely to local diffeomorphisms on $\mathcal{G}_k^{e,\bullet}$. If $(\widetilde{\mathcal{G}_k^{e,\bullet}}, \varepsilon \mathcal{G}_k)$ is globalizable, then we get the transformation group G which acts *simply transitively* on $\mathcal{G}_k^{e,\bullet}$ and $\mathcal{G}_k^{e,\bullet} \simeq G$ as before. It is extremely crucial to observe that the action of G may not descend to M , i.e., G may not act on M : for this we need the stronger condition $\mathcal{R}_k = 0$ which implies $\mathcal{R}_0 = 0$.

Now the algebroid of $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathcal{G}_k)$ is $\varepsilon \mathfrak{G}_k \rightarrow \mathcal{G}_k^{e,\bullet}$ which is simply $T(\mathcal{G}_k^{e,\bullet}) \rightarrow \mathcal{G}_k^{e,\bullet}$ together with the splitting $\varepsilon : T(\mathcal{G}_k^{e,\bullet}) \rightarrow \varepsilon T(\mathcal{G}_k^{e,\bullet}) \subset J_1 T(\mathcal{G}_k^{e,\bullet})$. We will denote this algebroid by $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ for notational convenience below. Let \mathfrak{R}_0 be the curvature of $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ obtained by linearizing \mathcal{R}_0 . For $\bar{p} \in \mathcal{G}_k^{e,\bullet}$ and $\xi_p, \eta_p \in \mathfrak{G}_k^p = T(\mathcal{G}_k^{e,\bullet})^{\bar{p}}$, we have

$$\mathfrak{R}_0(\bar{p})(\xi_p, \eta_p) \in Hom(\mathfrak{G}_k^p, \mathfrak{G}_k^p) \quad (50)$$

Clearly the P -algebra

$$\mathcal{P}^*(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathfrak{G}_k) \subset H_{dR}^*(\mathcal{G}_k^{e,\bullet}, \mathbb{R}) \quad (51)$$

vanishes since $\mathcal{G}_k^{e,\bullet}$ is parallelizable. We define the $2i$ -forms $Tr(\mathfrak{R}_0^i)$ on $\mathcal{G}_k^{e,\bullet}$ by

$$Tr(\mathfrak{R}_0^i)(\bar{p}, \xi_p^1, \xi_p^2, \dots, \xi_p^{2i}) \stackrel{def}{=} \frac{1}{(2k)!} \sum_{\sigma} sgn(\sigma) (Tr(\mathfrak{R}_0(\bar{p})(\xi_p^1, \xi_p^2) \circ \dots \circ \mathfrak{R}_0(\bar{p})(\xi_p^{2i-1}, \xi_p^{2i})) \quad (52)$$

where the summation is taken over all permutations σ of $(1, 2, \dots, 2j)$. The forms $Tr(\mathfrak{R}_0^i)$ are exact in the de Rham complex of $\mathcal{G}_k^{e,\bullet}$. In fact, the "Chern-Simons forms" with a surprisingly different interpretation supply some canonical primitives for $Tr(\mathfrak{R}_k^i)$ (see Section 10).

Now we have the following crucial

Lemma 25 $Tr(\mathfrak{R}_0^i)(\bar{p} \circ a) = Tr(\mathfrak{R}_0^i)(\bar{p})$ for all $a \in \mathcal{G}_k^{e,e}$ and $\bar{p} \in \mathcal{G}_k^{e,\bullet}$.

Lemma 25 states that $Tr(\mathfrak{R}_0^i)$ is a right invariant $2i$ -form on $\mathcal{G}_k^{e,\bullet} \rightarrow M$. Observe that $Tr(\mathfrak{R}_k^i)$ in Section 6 which generate $\mathcal{P}^*(M, \mathfrak{G}_k) \subset H_{dR}^*(M, \mathbb{R})$ also

live on $\mathcal{G}_k^{e,\bullet}$ but they are horizontal over M and therefore descend to M . However the forms $Tr(\mathfrak{R}_0^i)$ are surely not horizontal over M unless $k = 0$.

Now let $\wedge^k(\mathcal{G}_k^{e,\bullet}, M)$ denote the space of k -forms on $\mathcal{G}_k^{e,\bullet}$ which are right invariant over M . The exterior derivative d of the de Rham complex of $\mathcal{G}_k^{e,\bullet}$ restricts as $d : \wedge^k(\mathcal{G}_k^{e,\bullet}, M) \rightarrow \wedge^{k+1}(\mathcal{G}_k^{e,\bullet}, M)$ and we have the subcomplex

$$\wedge^0(\mathcal{G}_k^{e,\bullet}, M) \xrightarrow{d} \wedge^1(\mathcal{G}_k^{e,\bullet}, M) \xrightarrow{d} \dots \xrightarrow{d} \wedge^s(\mathcal{G}_k^{e,\bullet}, M) \quad (53)$$

where $s = \dim \mathcal{G}_k^{e,\bullet}$ and $\wedge^0(\mathcal{G}_k^{e,\bullet}, M) = C^\infty(\mathcal{G}_k^{e,\bullet})$. The cohomology $H_{inv}^*(\mathcal{G}_k^{e,\bullet}, M)$ of (53) is called the algebroid cohomology of $\mathfrak{G}_k \rightarrow M$ which we write also as $H^*(M, \mathfrak{G}_k)$. For $k = 0$ (53) is the de Rham complex of M but for $k \geq 1$ it is a proper subcomplex.

The crucial fact now is that the forms $Tr(\mathfrak{R}_0^i)$ which are exact in the de Rham complex of $\mathcal{G}_k^{e,\bullet}$ need not be exact in (53) for $k \geq 1$. Therefore the forms $Tr(\mathfrak{R}_0^i)$ generate a subalgebra $\widehat{\mathcal{P}}^*(M, \varepsilon \mathfrak{G}_k) \subset H_{inv}^{2i}(\mathcal{G}_k^{e,\bullet}, M) = H^*(M, \mathfrak{G}_k)$

Definition 26 *The algebra $\widehat{\mathcal{P}}^*(M, \varepsilon \mathfrak{G}_k) \subset H_{inv}^{2i}(\mathcal{G}_k^{e,\bullet}, M) = H^*(M, \mathfrak{G}_k)$ is the k -th order Pontryagin algebra of the phg $\varepsilon \mathfrak{G}_k$.*

Observe that $\widehat{\mathcal{P}}^*(M, \varepsilon \mathfrak{G}_k)$ depends on ε which is fixed by the definition of $\varepsilon \mathfrak{G}_k$. Clearly $\widehat{\mathcal{P}}^*(M, \varepsilon \mathfrak{G}_k) = 0$ for $k = 0$ since M is parallelizable.

The next proposition shows that the above construction of $(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ is a particular case.

Proposition 27 *The pair $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ defines a phg of order $k - i$ on the total space $\mathcal{G}_i^{e,\bullet}$ of the principal bundle $\mathcal{G}_i^{e,\bullet} \rightarrow M$ for $0 \leq i \leq k$.*

The main idea of Proposition 27 is simple: Let f be a local diffeomorphism on M with $f(p) = q$ and $j_i(f)^{p,q} \in \mathcal{G}_i^{p,q}$. Now f defines a function $\mathcal{G}_i^{e,p} \rightarrow \mathcal{G}_i^{e,q}$ by $\alpha_i^{e,p} \rightarrow j_i(f)^{p,q} \circ \alpha_i^{e,p}$. If $j_{i+1}(f)^{p,q} \in \mathcal{G}_{i+1}^{p,q}$ then $j_{i+1}(f)^{p,q}$ defines a 1-arrow from $\alpha_i^{e,p}$ to $j_i(f)^{p,q} \circ \alpha_i^{e,p}$ for any $\alpha_i^{e,p} \in \mathcal{G}_i^{e,p}$. Similarly $j_{i+2}(f)^{p,q} \in \mathcal{G}_{i+2}^{p,q}$ defines a 2-arrow from $\alpha_i^{e,p}$ to $j_i(f)^{p,q} \circ \alpha_i^{e,p}$. Iterating this process we see that $j_k(f)^{p,q} \in \mathcal{G}_k^{p,q}$ defines a $(k - i)$ -arrow from $\alpha_i^{e,p}$ to $j_i(f)^{p,q} \circ \alpha_i^{e,p}$. Finally, above any such $(k - i)$ -arrow, there is a unique $(k - i + 1)$ -arrow defined by $\varepsilon j_k(f)^{p,q} \in \varepsilon \mathcal{G}_k^{p,q}$.

Let \mathcal{R}_{k-i} denote the curvature of $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$. Now $\mathcal{R}_{k-i} = 0 \Leftrightarrow$ the pseudogroup $(\varepsilon \mathfrak{G}_k, \widetilde{\mathcal{G}_k^{e,\bullet}})$ which acts on $\mathcal{G}_k^{e,\bullet}$ descends to the pseudogroup $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ which acts on $\mathcal{G}_i^{e,\bullet}$. In particular, we have

Proposition 28 $\mathcal{R}_{k-i} = 0 \Rightarrow \mathcal{R}_{k-i-1} = \dots = \mathcal{R}_0 = 0$

The algebroid $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ of $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ is easily described explicitly: \mathfrak{G}_i^p is the tangent space at $\bar{p} \in \mathcal{G}_i^{e,p}$, \mathfrak{G}_{i+1}^p is 1-jets of vector fields at \bar{p} ...and \mathfrak{G}_k^p is the $(k - i)$ -jets of vector fields at \bar{p} . Finally, above any such $(k - i)$ -jet, there is a unique $(k - i + 1)$ -jet defined by $\varepsilon \mathfrak{G}_k^p$.

Let \mathfrak{R}_{k-i} be the curvature of $(\mathcal{G}_i^{e,\bullet}, \varepsilon \mathfrak{G}_k)$. We have

$$\mathfrak{R}_{k-i}(\bar{p}, \xi_p, \eta_p) \in \text{Hom}(\mathfrak{G}_k^p, \mathfrak{G}_k^p) \quad \bar{p} \in \mathcal{G}_i^{e, \bullet}, \quad \xi_p, \eta_p \in \mathfrak{G}_i^p \quad (54)$$

where $\bar{p} \in \mathcal{G}_i^{e, \bullet}$ projects to $p \in M$ and $\xi_p, \eta_p \in \mathfrak{G}_i^p = T(\mathcal{G}_i^{e, \bullet})^{\bar{p}}$. The algebra $\widehat{\mathcal{P}}^*(\mathcal{G}_0^{e, \bullet}, \varepsilon \mathfrak{G}_k) = \mathcal{P}^*(\mathcal{G}_0^{e, \bullet}, \varepsilon \mathfrak{G}_k)$ is considered in Section 6 and we defined $\widehat{\mathcal{P}}^*(\mathcal{G}_k^{e, \bullet}, \varepsilon \mathfrak{G}_k)$ above. Henceforth we assume $1 \leq i \leq k-1$.

Now using (54) and (52) we define the forms $Tr \mathfrak{R}_{k-i}^j$ in the de Rham complex of $\mathcal{G}_i^{e, \bullet}$ which are all exact because the principal bundle

$$\mathcal{G}_k^{e, \bullet} \rightarrow \mathcal{G}_i^{e, \bullet} \quad (55)$$

has contractible fibers and is therefore trivial. However, as in Lemma 25, we have

$$\mathfrak{R}_{k-i}(\bar{p}a) = \mathfrak{R}_{k-i}(\bar{p}) \quad (56)$$

where $a \in \mathcal{G}_i^{e, e}$ and we consider

$$\left[Tr \mathfrak{R}_{k-i}^j \right] \in H_{inv}^{2j}(\mathcal{G}_i^{e, \bullet}, M) = H_{inv}^*(\mathcal{G}_i^{e, \bullet}, M) \quad (57)$$

Definition 29 The subalgebra $\widehat{\mathcal{P}}^*(\mathcal{G}_i^{e, \bullet}, \varepsilon \mathfrak{G}_k) \subset H_{inv}^*(\mathcal{G}_i^{e, \bullet}, M)$ generated by the forms (54) is the i -th order Pontryagin algebra of $\varepsilon \mathfrak{G}_k$, $1 \leq i \leq k-1$.

Obviously $\widehat{\mathcal{P}}^*(\mathcal{G}_i^{e, \bullet}, \varepsilon \mathfrak{G}_k) = 0$ if $(\mathcal{G}_i^{e, \bullet}, \varepsilon \mathfrak{G}_k)$ is locally solvable.

Observe that there seems to be no reason for $\left[Tr \mathfrak{R}_{k-i}^j \right] = 0$ for j odd unless $i = 0$.

Unfortunately we do not get new invariants in Riemannian geometry because $H_{inv}^*(\mathcal{G}_1^{e, \bullet}, M) = H_{dR}^*(\mathcal{G}_1^{e, \bullet}, \mathbb{R})$ since $O(n)$ is compact.

The above method gives also obstructions to m -flatness as follows.

We have the groupoids $\mathcal{U}_{(k)} \stackrel{def}{=} J_{(k)}(M \times M)$ with the algebroids $J_{(k)}T \rightarrow M$, $k \geq 0$. There is a 1-1 correspondence between the following objects.

- i) Sections of $J_{(k+1)}T \rightarrow M$
- ii) Connections on the vector bundle $J_{(k)}T \rightarrow M$

Any such object ε_k defines a connection (using the same notation) ε_k on the tangent bundle $T\mathcal{U}_{(k)}^{e, \bullet} \rightarrow \mathcal{U}_{(k)}^{e, \bullet}$ where $\mathcal{U}_{(k)}^{e, \bullet} \rightarrow M$ is the principal bundle of the groupoid $\mathcal{U}_{(k)}$ with base point $e \in M$. We apply the Chern-Weil construction to ε_k and get the subalgebra $\mathcal{P}^*(\mathcal{U}_{(k)}^{e, \bullet}, T\mathcal{U}_{(k)}^{e, \bullet}) \subset H_{dR}^*(\mathcal{U}_{(k)}^{e, \bullet}, \mathbb{R})$ which is trivial. The forms obtained in this way are right invariant on $\mathcal{U}_{(k)}^{e, \bullet} \rightarrow M$ and a change of ε_k adds to such a form a boundary which is right invariant (compare to Proposition 31 below). Thus we get the subalgebra $\widehat{\mathcal{P}}^*(\mathcal{U}_{(k)}^{e, \bullet}, T\mathcal{U}_{(k)}^{e, \bullet}) \subset H_{inv}^*(\mathcal{U}_{(k)}^{e, \bullet}, M) = H^*(M, J_{(k)}T)$ = the cohomology of the algebroid $J_{(k)}T \rightarrow M$. Clearly $\widehat{\mathcal{P}}^*(\mathcal{U}_{(k)}^{e, \bullet}, T\mathcal{U}_{(k)}^{e, \bullet}) = 0$ if M is k -flat. This construction is the analog of Proposition 24.

More generally, we define

$$\widehat{\mathcal{P}}^*(\mathcal{U}_{(r)}^{e, \bullet}, J_{(k-r)}T\mathcal{U}_{(r)}^{e, \bullet}) \subset H_{inv}^*(\mathcal{U}_{(r)}^{e, \bullet}, M) = H^*(J_{(r)}T, M) \quad (58)$$

for $0 \leq r \leq k$ by observing that ε_k gives a connection on the vector bundle $J_{(k-r)}T\mathcal{U}_{(r)}^{e,\bullet} \rightarrow \mathcal{U}_{(r)}^{e,\bullet}$, $0 \leq r \leq k$. For $r = 0$ we get the topological obstructions $\widehat{\mathcal{P}}^*(M, J_{(k)}T) = \mathcal{P}^*(M, J_{(k)}T) \subset H_{dR}^*(M, \mathbb{R})$.

To summarize, we have

Proposition 30 *The restricted P-algebras $\widehat{\mathcal{P}}^*(\mathcal{U}_{(r)}^{e,\bullet}, J_{(k-r)}T\mathcal{U}_{(r)}^{e,\bullet}) \subset H_{inv}^*(\mathcal{U}_{(r)}^{e,\bullet}, M) = H^*(J_{(r)}T, M)$ for $0 \leq r \leq k$ vanish if M is k -flat.*

8 Dependence of the isomorphism class

Up to now we dealt with some fixed $\varepsilon\mathcal{G}_k$. Now we want to define the notion of equivalence of phg 's in such a way that the above constructions depend only on the equivalence class of $\varepsilon\mathcal{G}_k$. As a crucial point, we will not fix the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ and change the connection ε but change $\varepsilon\mathcal{G}_k$ and preserve the order of jets and the vertex class $\{\varepsilon\mathcal{G}_k\}$.

So we start with some $\varepsilon\mathcal{G}_k$ on M . Consider the group bundle $\mathcal{A}_{k+1} \stackrel{def}{=} \cup_{x \in M} \mathcal{U}_{k+1}^{x,x} \rightarrow M$. A smooth section of $\mathcal{A}_{k+1} \rightarrow M$ is called a gauge transformation of order $k+1$ on M . The set $\Gamma\mathcal{A}_{k+1}$ of gauge transformations is a group with fiberwise composition. Now $a \in \Gamma\mathcal{A}_{k+1}$ acts on the arrows of $\varepsilon\mathcal{G}_k$ by

$$a \cdot \varepsilon j_k(f)^{p,q} \stackrel{def}{=} a(q) \circ \varepsilon j_k(f)^{p,q} \circ a(p)^{-1} \quad (59)$$

We see that $a \cdot \varepsilon\mathcal{G}_k$ is another phg having the same vertex class as $\varepsilon\mathcal{G}_k$, i.e., $\{a \cdot \varepsilon\mathcal{G}_k\} = \{\varepsilon\mathcal{G}_k\}$. We have the projections $\pi : \mathcal{A}_{k+1} \rightarrow \mathcal{A}_i$, $1 \leq i \leq k$, which give the projections $\pi : \Gamma\mathcal{A}_{k+1} \rightarrow \Gamma\mathcal{A}_i$. By projecting (59) to the jets of order i , we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_k^{e,\bullet} & \rightarrow & a \cdot \mathcal{G}_k^{e,\bullet} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{G}_i^{e,\bullet} & \rightarrow & \pi(a) \cdot \mathcal{G}_i^{e,\bullet} \end{array} \quad (60)$$

where the horizontal arrows are isomorphisms of principal bundles.

The action of $\Gamma\mathcal{A}_{k+1}$ on $\varepsilon\mathcal{G}_k$ gives a natural action of $\Gamma\mathcal{A}_{k+1}$ on the algebroid $\varepsilon\mathfrak{G}_k$. We have the commutative diagram

$$\begin{array}{ccc} \varepsilon\mathcal{G}_k & \Longrightarrow & \varepsilon\mathfrak{G}_k \\ \downarrow & & \downarrow \\ a \cdot \varepsilon\mathcal{G}_k & \Longrightarrow & a \cdot \varepsilon\mathfrak{G}_k \end{array} \quad (61)$$

where \Longrightarrow denotes linearization. We write $\varepsilon\mathcal{G}_k \sim \varepsilon'\mathcal{G}'_k$ if $\varepsilon'\mathcal{G}'_k = a \cdot \varepsilon\mathcal{G}_k$ for some $a \in \Gamma\mathcal{A}_{k+1}$. We have $\varepsilon\mathcal{G}_k \sim \varepsilon'\mathcal{G}'_k \Leftrightarrow \varepsilon\mathfrak{G}_k \sim \varepsilon'\mathfrak{G}'_k$. We denote the equivalence classes of $\varepsilon\mathcal{G}_k$, $\varepsilon\mathfrak{G}_k$ by $[\varepsilon\mathcal{G}_k]$, $[\varepsilon\mathfrak{G}_k]$. Clearly the vector bundles $\varepsilon\mathfrak{G}_k \rightarrow M$ and $a \cdot \varepsilon\mathfrak{G}_k \rightarrow M$ are isomorphic. However the phg 's $\varepsilon\mathfrak{G}_k \rightarrow M$ and $\varepsilon'\mathfrak{G}'_k \rightarrow M$ may be isomorphic as vector bundles but inequivalent as phg 's as defined above. The main point is that the above equivalence respects the order of jets whereas the topological concept of "vector bundle isomorphism" does not.

Now the assignment $\varepsilon\mathfrak{G}_k \Rightarrow \mathcal{P}^*(\mathcal{G}_0^{e,\bullet}, \varepsilon\mathfrak{G}_k)$ is rather crude as it depends on the isomorphism class of the vector bundle $\varepsilon\mathfrak{G}_k \rightarrow M$. However, it turns out that the assignments $\varepsilon\mathfrak{G}_k \Rightarrow \widehat{\mathcal{P}}^*(\mathcal{G}_i^{e,\bullet}, \varepsilon\mathfrak{G}_k)$, $1 \leq i \leq k$ depend on $[\varepsilon\mathfrak{G}_k]$ in a sense to be made precise below.

Recall that $\mathcal{C}^*(\mathcal{G}_i^{e,\bullet}, M)$ denotes the complex of right invariant forms on the principal bundle $\mathcal{G}_i^{e,\bullet} \rightarrow M$, i.e., the complex computing the algebroid cohomology of $\mathfrak{G}_i \rightarrow M$. The linearization of the bottom isomorphism of (61) shows that $a \in \Gamma\mathcal{A}_{k+1}$ defines an isomorphism

$$a^* : \mathcal{C}^*(\mathcal{G}_i^{e,\bullet}, M) \rightarrow \mathcal{C}^*(\pi(a) \cdot \mathcal{G}_i^{e,\bullet}, M) \quad (62)$$

for $0 \leq i \leq k$. For $i = 0$, $a^* = Id$ and both complexes are the de Rham complex of M . It follows that a^* acts as an isomorphism on the complex (53) as

$$\begin{array}{ccccccc} \wedge^0(\mathcal{G}_i^{e,\bullet}, M) & \xrightarrow{d} & \wedge^1(\mathcal{G}_i^{e,\bullet}, M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \wedge^s(\mathcal{G}_i^{e,\bullet}, M) \\ \downarrow a^* & & \downarrow a^* & & \downarrow a^* & & \downarrow a^* \\ \wedge^0(a \cdot \mathcal{G}_i^{e,\bullet}, M) & \xrightarrow{d} & \wedge^1(a \cdot \mathcal{G}_i^{e,\bullet}, M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \wedge^s(a \cdot \mathcal{G}_i^{e,\bullet}, M) \end{array} \quad (63)$$

Therefore a^* induces an isomorphism

$$a^* : H^*(\mathcal{G}_i^{e,\bullet}, M) \rightarrow H^*(a \cdot \mathcal{G}_i^{e,\bullet}, M) \quad (64)$$

Consider the forms $Tr\mathfrak{R}_{k-i}^j \in \wedge^{2j}(\mathcal{G}_i^{e,\bullet}, M)$. Let $Tr\mathfrak{R}_{k-i}^j(a) \in \wedge^{2j}(a \cdot \mathcal{G}_i^{e,\bullet}, M)$ denote the forms constructed on $a \cdot \varepsilon\mathcal{G}_k$ using the curvature $\mathfrak{R}_k(a)$ of $(a \cdot \mathcal{G}_i^{e,\bullet}, M)$ in the same way. Thus $a^*Tr\mathfrak{R}_{k-i}^j$ and $Tr\mathfrak{R}_{k-i}^j(a)$ both live in $\wedge^{2j}(M, a \cdot \mathcal{G}_i^{e,\bullet})$.

Now we have the following fundamental

Proposition 31 $a^*Tr\mathfrak{R}_{k-i}^j - Tr\mathfrak{R}_{k-i}^j(a)$ is exact in the bottom complex of (63).

Corollary 32 a^* induces an isomorphism

$$a^* : \widehat{\mathcal{P}}^*(\mathcal{G}_i^{e,\bullet}, \varepsilon\mathfrak{G}_k) \longrightarrow \widehat{\mathcal{P}}^*(\pi(a) \cdot \mathcal{G}_i^{e,\bullet}, a \cdot \varepsilon\mathfrak{G}_k) \quad (65)$$

Corollary 33 We have the well defined assignments

$$[\varepsilon\mathcal{G}_k] \Longrightarrow \widehat{\mathcal{P}}^*(\mathcal{G}_i^{e,\bullet}, \varepsilon\mathfrak{G}_k) \quad 0 \leq i \leq k \quad (66)$$

In particular, $\widehat{\mathcal{P}}^*(\mathcal{G}_i^{e,\bullet}, \varepsilon\mathfrak{G}_k) = 0$, $0 \leq i \leq k$, if the equivalence class $[\varepsilon\mathcal{G}_k]$ contains a locally solvable phg.

If $U \subset M$ is an open subset, we define the restriction $\varepsilon\mathcal{G}_{k|U}$ of $\varepsilon\mathcal{G}_k$ as the arrows whose source and targets are contained in U . Clearly $\varepsilon\mathcal{G}_{k|U}$ also satisfies i), ii) of Definition 1 and therefore defines a phg on U . By Corollary 33 we obtain the algebras $\widehat{\mathcal{P}}^*(\mathcal{G}_{i|U}^{e,\bullet}, \varepsilon\mathfrak{G}_{k|U})$.

Proposition 34 $\widehat{\mathcal{P}^*}(\mathcal{G}_{i|U}^{e,\bullet}, \varepsilon \mathcal{G}_{k|U}) = 0$ for a coordinate neighborhood $U \subset M$ diffeomorphic to \mathbb{R}^n .

To see this, *iii*) of Definition 1 implies $\langle \varepsilon \mathcal{G}_k \rangle = \langle H \rangle_G$ for some homogeneous space $G/H = N$. Let $\varepsilon \mathcal{G}'_k$ be the globally solvable phg on N defined by G/H and consider $\varepsilon' \mathcal{G}'_{k|V}$ for some $V \subset N$ diffeomorphic to \mathbb{R}^n which gives an identification $U \simeq V$. So we have the two phg 's $\varepsilon \mathcal{G}'_{k|V}$ and $\varepsilon' \mathcal{G}'_{k|V}$ defined on V and $\varepsilon' \mathcal{G}'_{k|V}$ is locally solvable. We claim that they are equivalent: Since the vertex groups $\varepsilon \mathcal{G}_{k|V}^{p,p}$ and $\varepsilon \mathcal{G}_{k|V}^{p,p}$ are conjugate inside $\mathcal{U}_{k+1}^{p,p}$ for all $p \in V$, there exists some $a(p) \stackrel{def}{=} a_{k+1}^{p,p} \in \mathcal{U}_{k+1}^{p,p}$ with $a(p) \cdot \varepsilon \mathcal{G}_{k+1|V}^{p,p} = \varepsilon' \mathcal{G}'_{k+1|V}^{p,p}$. Let $\mathcal{A}(p)$ denote the set of all such $a(p)$'s. The set $\mathcal{A}(p)$ is in 1-1 correspondence with the normalizer of $\mathcal{G}_{k+1}^{p,p}$ inside $\mathcal{U}_{k+1}^{p,p}$. We now have the bundle $\cup_{p \in U} \mathcal{A}(p) \rightarrow U$ which admits a crossection.

We will conclude this section with three remarks.

1) We defined a Riemann geometry $\varepsilon \mathcal{G}_1$ in Example 3 using the pair (g, ε) where ε is the LC-connection of g . Suppose $\varepsilon \mathcal{G}_1 \sim \varepsilon' \mathcal{G}'_1$ so that $\varepsilon' \mathcal{G}'_1$ is defined by the geometric object (g', ε') . Now ε' need not be the LC connection of g' ! The reason is the Christoffel symbols ε_{jk}^i can be expressed in terms of the derivatives of g_{ij} whereas a gauge transformation is a section of jets and preserves differentiation only pointwise at the level of jets but not locally. Now for $p \in M$, we can find a coordinate system around p such that $g_{ij} = \delta_{ij}$ and $\varepsilon_{jk}^i = 0$ at p . The components ε_{jk}^i of ε' also satisfy this condition at p but we may not be able to express ε_{jk}^i locally in terms of the derivatives of g'_{ij} .

Now there exists a somewhat stronger concept of equivalence of phg 's which, in view of the proof of Proposition 34, exhibits a unique canonical representative in each equivalence class which is "defined in terms of the derivatives of some geometric object" and in particular gives the Levi-Civita connection in Riemann geometry. This gives a generalization of the main construction of [7] for parabolic geometries to all phg 's. The idea is simple: for any phg $\varepsilon \mathcal{G}_k$ we can define a geometric object \mathbf{g} of order $k+1$ on M such that a $(k+1)$ -arrow of \mathcal{U}_{k+1} belongs to $\varepsilon \mathcal{G}_k$ if and only if it preserves \mathbf{g} . Therefore this condition gives the defining equations of $\varepsilon \mathcal{G}_k$. It is very easy to construct \mathbf{g} : For $x \in M$ consider the left coset space $\mathcal{U}_{k+1}^{x,x} / \varepsilon \mathcal{G}_k^{x,x}$ and define the bundle of geometric objects $\mathcal{O} \stackrel{def}{=} \cup_{x \in M} \mathcal{U}_{k+1}^{x,x} / \varepsilon \mathcal{G}_k^{x,x} \rightarrow M$. Now $\varepsilon \mathcal{G}_k$ defines a global crossection of $\mathcal{O} \rightarrow M$ which is \mathbf{g} . In Example 3 $\mathbf{g} = (g, \varepsilon)$ and in Example 4 $\mathbf{g} = S$ = the expression for the Schwarzian derivative! Now $a \in \Gamma \mathcal{A}_{k+1}$ acts on the sections of $\mathcal{O} \rightarrow M$ on the left. If $a * \mathbf{g} = \mathbf{g}'$ then \mathbf{g}' defines another phg $a * \varepsilon \mathcal{G}_k$ which preserves \mathbf{g}' and is equivalent to $\varepsilon \mathcal{G}_k$ as defined by (59) because the left coset $\beta \varepsilon \mathcal{G}_k^{x,x}$ defines the conjugate $\beta \varepsilon \mathcal{G}_k^{x,x} \beta^{-1}$ but not conversely since the normalizer of $\varepsilon \mathcal{G}_k^{x,x}$ inside $\mathcal{U}_{k+1}^{x,x}$ may strictly contain $\varepsilon \mathcal{G}_k^{x,x}$ in general. In short, the philosophy of (59) is to preserve the symmetry group of the object whereas the philosophy of the second is to preserve the object itself.

2) Let $\Gamma \mathcal{A}_{k+1,i}$ denote the group of sections of $\mathcal{A}_{k+1,i} \rightarrow M$ defined as the kernel of the projection $\pi : \mathcal{A}_{k+1} \rightarrow \mathcal{A}_i$. If $a \in \Gamma \mathcal{A}_{k+1,i}$, then $\varepsilon \mathcal{G}_k$ and $a \cdot \varepsilon \mathcal{G}_k$

define the same principal bundle $\mathcal{G}_i^{e,\bullet} \rightarrow M$ since $\pi(a) = Id$. Thus we get the isomorphism of algebroids $a^* : (\mathcal{G}_i^{e,\bullet}, \varepsilon\mathfrak{G}_k) \longrightarrow (\mathcal{G}_i^{e,\bullet}, a \cdot \varepsilon\mathfrak{G}_k)$ of order $k - i$. In particular, let us choose $i = k$. Recalling that $\varepsilon : \mathcal{G}_k \rightarrow \varepsilon\mathcal{G}_k$ is a connection on the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$, acting with $a \in \Gamma\mathcal{A}_{k+1,k}$ on $\varepsilon\mathcal{G}_k$ amounts to fixing $\mathcal{G}_i^{e,\bullet}$ but changing the connection. So the actions of $a \in \Gamma\mathcal{A}_{k+1,k}$ give points on the moduli space $\mathcal{M}(\mathcal{G}_k^{e,\bullet})$ of connections on the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ as defined in gauge theory whereas they keep us inside $[\varepsilon\mathcal{G}_k]$. It follows that $[\varepsilon\mathcal{G}_k]$ plays the role of $\mathcal{M}(\mathcal{G}_k^{e,\bullet})$ and (66) reflects the philosophy of attaching certain invariants to $\mathcal{M}(\mathcal{G}_k^{e,\bullet})$.

3) All the constructions in this note can be done in the holomorphic category by considering k -jets of holomorphic objects and then working in the smooth category as above. However many subtleties arise. For instance, the standard definition of $\mathcal{P}^*(M, \varepsilon\mathfrak{G}_k)$ in terms of the Chern classes of the complexification of $\varepsilon\mathfrak{G}_k \rightarrow M$ becomes rather artificial because the underlying idea is the complexification of a homogeneous space which is a nontrivial problem even for Lie groups (see [18], pg. 429-430 and [11]).

9 Appendix A: Cartan connections

We resume the setting of Section 7. Suppose $\mathcal{R}_0 = 0 \Leftrightarrow \mathfrak{R}_0 = 0$ so that the phg $(\mathcal{G}_k^{e,\bullet}, \varepsilon\mathcal{G}_k)$ of order zero integrates to the pseudogroup $(\widetilde{\mathcal{G}_k^{e,\bullet}}, \varepsilon\mathcal{G}_k)$. Therefore $\widehat{\mathcal{P}^*}(\mathcal{G}_k^{e,\bullet}, \varepsilon\mathfrak{G}_k) = 0$. For simplicity, assume that $(\mathcal{G}_k^{e,\bullet}, \varepsilon\mathcal{G}_k)$ globalizes to a Lie group G so that G acts simply transitively on $\mathcal{G}_k^{e,\bullet}$ and $\mathcal{G}_k^{e,\bullet} \simeq G$. Recall that the action of G may not descend to $\mathcal{G}_{k-1}^{e,\bullet}$ since we may not have $\mathfrak{R}_1 = 0$. Therefore, recalling (11), we see that the assumption $\mathcal{R}_0 = 0$ fixes also the group other than the vertex class in the definition of the phg . Let \mathfrak{g} denote the Lie algebra of the infinitesimal generators of G which can be identified with a subalgebra of $\mathfrak{X}(\mathcal{G}_k^{e,\bullet})$ = the Lie algebra of vector fields on $\mathcal{G}_k^{e,\bullet}$. Recall that this identification is done by evaluating the infinitesimal generators at some point and therefore is not canonical.

Now the restriction of D'_1 in (37) to the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ defines an \mathfrak{h} -valued 1-form on $\mathcal{G}_k^{e,\bullet}$ where \mathfrak{h} = the Lie algebra of $\mathcal{G}_k^{e,e}$. This is the connection in Proposition 18. If also $\mathfrak{R}_0 = 0$ as we now assume, we can define a \mathfrak{g} -valued 1-form ω on $\mathcal{G}_k^{e,\bullet}$ as follows. Let $\bar{p} = j_k(f)^{e,p} \in \mathcal{G}_k^{e,\bullet}$ and $\xi_{\bar{p}} \in T(\mathcal{G}_k^{e,\bullet})^{\bar{p}} = \mathfrak{G}_k^p \simeq \mathfrak{g}$. We rewrite (26) as

$$\varepsilon j_k(f^{-1})_*^{p,e} : T_{\bar{p}}(\mathcal{G}_k^{e,\bullet})^{\bar{p}} \rightarrow T(\mathcal{G}_k^{e,\bullet})^{\bar{e}} = \mathfrak{g} \quad (67)$$

We define $\omega \stackrel{def}{=} \varepsilon j_k(f^{-1})_*^{p,e}$ and easily show

Proposition 35 ω is a Cartan connection on $\mathcal{G}_k^{e,\bullet} \rightarrow M$.

Let $\overline{\mathfrak{R}}$ denote the curvature of ω which is a \mathfrak{g} -valued 2-form on $\mathcal{G}_k^{e,\bullet}$. Since the conditions $\overline{\mathfrak{R}} = 0$ and $\mathfrak{R}_k = 0$ are both equivalent to local homogeneity of M , obviously we have

Proposition 36 $\overline{\mathfrak{R}} = 0 \Leftrightarrow \mathfrak{R}_k = 0$

Now $\mathfrak{R}_k(\overline{p})(\eta_p, \sigma_p) : \mathfrak{G}_k^p \rightarrow \mathfrak{G}_k^p$ becomes

$$\mathfrak{R}_k(\overline{p})(\eta_p, \sigma_p) : \mathfrak{g} \rightarrow \mathfrak{g} \quad (68)$$

Proposition 37 (68) is a derivation.

Proposition 37 together with the interpretation of $Der(\mathfrak{g})$ in [15] now gives a very interesting interpretation of \mathfrak{R}_k .

We recall the representation

$$ad : \mathfrak{g} \rightarrow Der(\mathfrak{g}) \quad (69)$$

and assume

A2: (69) is an isomorphism.

For instance **A2** holds if \mathfrak{g} is semisimple. However, this assumption forces $k \leq 2$ and all our efforts with higher order jets fall flat! Assuming **A2**, we identify $\mathfrak{R}_k(\overline{p})(\eta_p, \sigma_p)$ uniquely with an element of \mathfrak{g} so that \mathfrak{R}_k becomes a \mathfrak{g} -valued 2-form on $\mathcal{G}_k^{e,\bullet} \rightarrow M$.

Proposition 38 If **A2** holds, then $\overline{\mathfrak{R}} = \mathfrak{R}_k$.

Thus we conclude that \mathfrak{R}_0 and therefore $\widehat{\mathcal{P}}^*(\mathcal{G}_k^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ is an obstruction to the existence of a Cartan connection on the principal bundles $a \cdot \mathcal{G}_k^{e,\bullet} \rightarrow M$, $a \in \Gamma \mathcal{A}_{k+1}$.

Assuming $\mathfrak{R}_0 = 0$, now \mathfrak{R}_1 and therefore $\widehat{\mathcal{P}}^*(\mathcal{G}_{k-1}^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ is an obstruction to the existence of a (generalized) Cartan connection on the principal bundles $\pi_{k,k-1}(a) \cdot \mathcal{G}_{k-1}^{e,\bullet} \rightarrow M$, $a \in \Gamma \mathcal{A}_{k+1}$ in a way which is straightforward at this stage. With the assumptions $\mathfrak{R}_i = 0$, $0 \leq i \leq k-1$, we finally encounter the topological obstructions $\widehat{\mathcal{P}}^*(\mathcal{G}_0^{e,\bullet}, \varepsilon \mathfrak{G}_k)$ to the homogeneity of M by the action of G .

10 Appendix B: Chern-Simons forms

In this section $\varepsilon \mathcal{G}_k$ is a phg on M with $k = 0$. Equivalently, we have a splitting $\varepsilon : M \times M \rightarrow \mathcal{U}_1$ which in turn is equivalent to the parallelizability of M . We refer to [1] for a detailed study of this case. According to Proposition 24, the total space of the principal bundle $\mathcal{G}_k^{e,\bullet} \rightarrow M$ defined by $\varepsilon \mathcal{G}_k$ is parallelizable which gives a rich source of examples.

Now since M is parallelized by ε , $\widehat{\mathcal{P}}^*(\mathcal{G}_0^{e,\bullet}, \varepsilon \mathfrak{G}_0) = \mathcal{P}^*(M, T) = 0$ and therefore the forms $\mathfrak{R}_0^{2i} \in \wedge^{4i} T^*$ defined by (52) are exact. Our purpose here is to show that the "Chern-Simons" forms (but with a surprisingly different interpretation) furnish some canonical primitives of these forms. Henceforth we denote the curvature \mathfrak{R}_0 by \mathfrak{R} .

First, we recall from [1] the definition of the curvature $\widetilde{\mathfrak{R}}$ defined by

$$\tilde{\mathfrak{R}}_{rj,k}^i = \left[\frac{\partial \Gamma_{jk}^i}{\partial x^r} + \Gamma_{rk}^a \Gamma_{ja}^i \right]_{[rj]} \quad (70)$$

where

$$\Gamma_{jk}^i(x) \stackrel{def}{=} \left[\frac{\partial \varepsilon_k^i(x,y)}{\partial y^j} \right]_{y=x} \quad (71)$$

We always have $\tilde{\mathfrak{R}} = 0$ on a parallelizable manifold (M, ε) . The reason is that $\tilde{\mathfrak{R}} = 0$ gives the integrability conditions of

$$\tilde{\nabla}_r \xi^i \stackrel{def}{=} \frac{\partial \xi^i}{\partial x^r} - \Gamma_{ra}^i \xi^a = 0 \quad (72)$$

and a vector field $\xi = (\xi^i)$ solves (72) if and only if it is ε -invariant. Since we start with the global parallelism ε , we can always construct ε -invariant vector fields with arbitrary initial conditions and therefore $\tilde{\mathfrak{R}} = 0$. However

$$\mathfrak{R}_{rj,k}^i = \left[\frac{\partial \Gamma_{kj}^i}{\partial x^r} + \Gamma_{kr}^a \Gamma_{aj}^i \right]_{[rj]} \quad (73)$$

and $\mathfrak{R} = 0$ if and only if M is locally homogeneous in which case (M, ε) is called a local Lie group in [1].

Using $\tilde{\mathfrak{R}} = 0$, we will now construct a locally exact complex. Consider the vector bundle $T^* \otimes T \rightarrow M$ isomorphic to $Hom(T, T) \rightarrow M$ and the vector bundle $\wedge^k T^* \otimes Hom(T, T) \rightarrow M$, the bundle of k -forms on M with values in $Hom(T, T)$. A local section of $\wedge^k T^* \otimes Hom(T, T) \rightarrow M$ is of the form $\omega_{h_k \dots h_1, j}^i$ where ω is alternating in the indices h_k, \dots, h_1 . We define the local operator \tilde{d}_r by the formula

$$\tilde{d}_r \omega_{h_k \dots h_1, j}^i \stackrel{def}{=} \frac{\partial \omega_{h_k \dots h_1, j}^i}{\partial x^r} - \Gamma_{ra}^i \omega_{h_k \dots h_1, j}^a + \Gamma_{rj}^a \omega_{h_k \dots h_1, a}^i \quad (74)$$

The operator \tilde{d}_r has a coordinatefree meaning only for $k = 0$ in which case $\tilde{d}_r = \tilde{\nabla}_r$ which is defined as an extension of (72) on arbitrary tensor fields. Now we define the first order linear differential operator

$$\tilde{d} : \wedge^k T^* \otimes Hom(T, T) \longrightarrow \wedge^{k+1} T^* \otimes Hom(T, T) \quad (75)$$

by the formula

$$\begin{aligned} & \left(\tilde{d}\omega \right)_{rh_k \dots h_1, j}^i \stackrel{def}{=} \left[\tilde{d}_r \omega_{h_k \dots h_1, j}^i \right]_{[rh_k \dots h_1]} \\ &= \tilde{d}_r \omega_{h_k \dots h_1, j}^i - \tilde{d}_{h_k} \omega_{rh_{k-1} \dots h_1, j}^i - \dots - \tilde{d}_{h_1} \omega_{rh_k \dots h_2, j}^i \end{aligned} \quad (76)$$

Since $\tilde{\mathfrak{R}} = 0$, we have $\tilde{d} \circ \tilde{d} = 0$ and we obtain the complex

$$Hom(T, T) \xrightarrow{\tilde{d}} \wedge^1 T^* \otimes Hom(T, T) \xrightarrow{\tilde{d}} \dots \xrightarrow{\tilde{d}} \wedge^n T^* \otimes Hom(T, T) \quad (77)$$

which is locally exact. It is easy to give a coordinate free description of (77) which is a well known construction. However, observe that (77) is not a complex with the accordingly defined operators d_r since we do not assume $\mathfrak{R} = 0$ (see (93) below). The kernel $\widetilde{Hom(T, T)}$ of the first operator in (77) is ε -invariant sections of $Hom(T, T) \rightarrow M$ and (77) is a fine resolution of the sheaf $\widetilde{Hom(T, T)}$.

Now let $\omega \in \wedge^k T^* \otimes Hom(T, T)$ and $\psi \in \wedge^m T^* \otimes Hom(T, T)$. We define $\omega \wedge \psi$ by the formula

$$(\omega \wedge \psi)_{h_k \dots h_1 s_m \dots s_1, j}^i \stackrel{def}{=} [\omega_{h_k \dots h_1, a}^i \psi_{s_m \dots s_1, j}^a]_{[h_k \dots h_1 s_m \dots s_1]} \quad (78)$$

or in coordinatefree form $(\omega \wedge \psi)(X_1, \dots, X_k, X_{k+1}, \dots, Y_{k+m}) \stackrel{def}{=}$

$$\sum_{\sigma \in S_{k+m}} \frac{1}{k!m!} sgn(\sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \circ \psi(X_{\sigma(k+1)}, \dots, Y_{\sigma(k+m)}) \quad (79)$$

where \circ denotes composition in $Hom(T, T)$. We have

$$\tilde{d}(\omega \wedge \psi) = (\tilde{d}\omega) \wedge \psi + (-1)^{\deg(\omega)} \omega \wedge (\tilde{d}\psi) \quad (80)$$

We recall the definition

$$T_{k,j}^i \stackrel{def}{=} \Gamma_{kj}^i - \Gamma_{jk}^i \quad (81)$$

Now $T = (T_{k,j}^i) \in \wedge^1 T^* \otimes Hom(T, T)$ and $\mathfrak{R} = (\mathfrak{R}_{km,j}^i) \in \wedge^2 T^* \otimes Hom(T, T)$.

We have the following fundamental

Proposition 39 (*Structure equation*)

$$\tilde{d}T + T \wedge T = \mathfrak{R} \quad (82)$$

It is worthwhile to observe the remarkable analogy between (82) and the well known structure equation

$$dA + A \wedge A = R \quad (83)$$

on a principal bundle $P \rightarrow M$ where d is the exterior derivative, A the Lie algebra valued connection 1-form and R its curvature 2-form. Observe that (82) is defined on M whereas (83) is defined on P . To make this analogy more precise, let $P \rightarrow M$ be $\mathcal{U}_1^{e, \bullet} \rightarrow M$ with structure group $\mathcal{U}_1^{e, \bullet} \simeq GL(n, \mathbb{R})$ with Lie algebra $\mathfrak{h} = \mathfrak{gl}(n, \mathbb{R})$. Now Γ_{jk}^i defined in terms of the absolute parallelism ε by the formula (71) transform as the components of $\mathfrak{gl}(n, \mathbb{R})$ -valued 1-form on $\mathcal{U}_1^{e, \bullet} \rightarrow M$ with curvature R . In this particular case $Hom(T, T) \simeq \mathfrak{gl}(n, \mathbb{R})$ (see the last paragraph of Section 4) and therefore R and \mathfrak{R} live in the same

space...but they are again different because $R = \tilde{\mathfrak{R}} = 0$ whereas \mathfrak{R} need not vanish.

Proposition 39 shows that \mathfrak{R} is determined by T . In fact, we have the following fundamental

Proposition 40

$$\tilde{\nabla}_r T_{jk}^i = \mathfrak{R}_{jk,r}^i \quad (84)$$

Therefore $\mathfrak{R} = 0 \Leftrightarrow T$ is ε -invariant. Observe that \mathfrak{R} and T have the same alternating indices j, k .

Now let $\omega \in \wedge^k T^* \otimes \text{Hom}(T, T)$. We define $Tr(\omega) \in \wedge^k T^*$ by the formula

$$Tr(\omega_{h_k \dots h_1, j}^i) \stackrel{def}{=} \omega_{h_k \dots h_1, a}^a \quad (85)$$

So we obtain the following commutative diagram

$$\begin{array}{ccccccc} \text{Hom}(T, T) & \xrightarrow{\tilde{d}} & \wedge^1 T^* \otimes \text{Hom}(T, T) & \xrightarrow{\tilde{d}} & \dots & \xrightarrow{\tilde{d}} & \wedge^n T^* \otimes \text{Hom}(T, T) \\ \downarrow Tr & & \downarrow Tr & & \downarrow Tr & & \downarrow Tr \\ C^\infty(M) & \xrightarrow{d} & \wedge^1 T^* & \xrightarrow{d} & \dots & \xrightarrow{d} & \wedge^n T^* \end{array} \quad (86)$$

where the lower complex in (86) is the de Rham complex of M .

With the notation (52), we now have

$$\mathfrak{R}^i = \mathfrak{R} \wedge \mathfrak{R} \wedge \dots \wedge \mathfrak{R} \quad (i\text{-copies}) \quad (87)$$

and we define

$$T^i \stackrel{def}{=} T \wedge T \wedge \dots \wedge T \quad (i\text{-copies}) \quad (88)$$

Clearly $\mathfrak{R}^i \in \wedge^{2i} T^* \otimes \text{Hom}(T, T)$ and $T^i \in \wedge^i T^* \otimes \text{Hom}(T, T)$. Therefore $Tr(\mathfrak{R}^i) \in \wedge^{2i} T^*$ and $Tr(T^i) \in \wedge^i T^*$. It is easy to see that

$$Tr(T^{2i}) = 0 \quad (89)$$

and we are left with $Tr(T^{2i+1})$, $i \geq 0$. Omitting \wedge from our notation, applying \tilde{d} to (82) and substituting back from (82) we obtain

$$\tilde{d}\mathfrak{R} = \mathfrak{R}T - T\mathfrak{R} \quad (90)$$

Using (82), (90) and (80) we compute

$$\begin{aligned} \tilde{d}(T^3) &= \mathfrak{R}T^2 - T\mathfrak{R}T + T^2\mathfrak{R} - T^4 \\ \tilde{d}(\mathfrak{R}T) &= -T\mathfrak{R}T + \mathfrak{R}^2 \end{aligned} \quad (91)$$

Taking the trace of the formulas in (91) and observing $Tr(T\mathfrak{R}T) = -Tr(\mathfrak{R}T^2) = -Tr(T^2\mathfrak{R})$, we deduce

$$dTr\left(\Re T - \frac{1}{3}T^3\right) = Tr(\Re^2) \quad (92)$$

Observe the "Chern-Simons" 3-form in (92) with the surprising difference that the Lie algebra valued 1-form A in (83) is replaced with the $Hom(T, T)$ -valued 1-form T and it lives on the base M !! The higher degree Chern-Simons forms are derived in the same way without any further computation but as a logical consequence of the correspondence between (82) and (83).

To complete the analogy to the formalism of connections on principal bundles, we recall the Bianchi identity

$$DR = 0 \quad (93)$$

on $P \rightarrow M$ where D is the exterior covariant differentiation. We define

$$d_r \omega_{h_k \dots h_1, j}^i \stackrel{def}{=} \frac{\partial \omega_{h_k \dots h_1, j}^i}{\partial x^r} - \Gamma_{ar}^i \omega_{h_k \dots h_1, j}^a + \Gamma_{jr}^a \omega_{h_k \dots h_1, a}^i \quad (94)$$

and the operator

$$d : \wedge^k T^* \otimes Hom(T, T) \longrightarrow \wedge^{k+1} T^* \otimes Hom(T, T) \quad (95)$$

by the formula (76) using d_r instead of \tilde{d}_r . Observe that

$$d_r \omega_{h_k \dots h_1, j}^i = \tilde{d}_r \omega_{h_k \dots h_1, j}^i - T_{ar}^i \omega_{h_k \dots h_1, j}^a + T_{rj}^a \omega_{h_k \dots h_1, a}^i \quad (96)$$

Now $d \circ d \neq 0$ because \Re is the obstruction to $d \circ d = 0$.

The analog of (83) is

Proposition 41 (*Bianchi identity*) $d\Re = 0$

Finally, the first formula in (91) shows $dTr(T^3) = 0$ if $\Re = 0$. An easy induction gives

Proposition 42 *If $\Re = 0$, then $dTr(T^{2i+1}) = 0$.*

Definition 43 $[Tr(T^{2i+1})] \in H_{dR}^{2i+1}(M, \mathbb{R})$ are the secondary characteristic classes of the local Lie group (M, ε) .

The secondary characteristic classes coincide with the Chern-Simons classes on a local Lie group.

We will conclude with a question. Recall that the forms $Tr(\Re_{k-i}^j) \in H_{dR}^{2j}(\varepsilon \mathcal{G}_i^{e, \bullet}, \mathbb{R})$ are exact for $1 \leq i \leq k$ and for $i = k$ we found above some explicit primitives as "Chern-Simons" forms.

Q : Find some explicit primitives for $1 \leq i \leq k - 1$.

11 Appendix C. Uniformization number and representations

Let $(\mathfrak{g}, \mathfrak{h})$ be a Lie algebra pair with $\mathfrak{h} \subset \mathfrak{g}$. We set $V = \mathfrak{g}/\mathfrak{h}$ and define $ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}} : \mathfrak{h} \rightarrow gl(V)$ by

$$ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}}(h)(g + \mathfrak{h}) \stackrel{def}{=} [h, g] + \mathfrak{h} \quad (97)$$

Clearly $ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}}$ is well defined and is a representation of \mathfrak{h} .

Definition 44 $ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}}$ is the adjoint representation of \mathfrak{h} relative to $\mathfrak{g}/\mathfrak{h}$.

Suppose $(\mathfrak{g}', \mathfrak{h})$ is another such pair. We call $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h})$ isomorphic and write $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}', \mathfrak{h})$ if $\mathfrak{g} \simeq \mathfrak{g}'$ and the isomorphism \simeq restricts to identity on \mathfrak{h} . We easily check that if $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}', \mathfrak{h})$ then the representations $ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}}$ and $ad_{\mathfrak{h}, \mathfrak{g}'/\mathfrak{h}}$ are isomorphic but not conversely.

Proposition 45 Any representation $\rho : \mathfrak{h} \rightarrow gl(W)$ is an adjoint representation relative to some $\mathfrak{g} \supset \mathfrak{h}$.

Indeed, given a representation $\rho : \mathfrak{h} \rightarrow gl(W)$, we set $\mathfrak{g} \stackrel{def}{=} \mathfrak{h} \times W$ and check that \mathfrak{g} is a Lie algebra with the bracket defined by $[(h, w), (h', w')] \stackrel{def}{=} ([h, h'], \rho(h)w' - \rho(h')w)$. We identify \mathfrak{h} with the subalgebra $(\mathfrak{h}, 0) \subset \mathfrak{g}$ and W with $\mathfrak{g}/\mathfrak{h}$ and check that $\rho = ad_{\mathfrak{h}, \mathfrak{g}/\mathfrak{h}}$ with these identifications.

Therefore adjoint representations exhaust all representations! From the above construction of the pair $(\mathfrak{h} \times W, \mathfrak{h})$ we deduce

Proposition 46 Let \mathfrak{h} be a (finite dimensional) Lie algebra. The following are equivalent.

- i) (Ado's Theorem) \mathfrak{h} has a faithful representation
- ii) There exists a (finite dimensional) Lie algebra $\mathfrak{g} \supset \mathfrak{h}$ such that the pair $(\mathfrak{g}, \mathfrak{h})$ is effective.

Indeed, if $\rho : \mathfrak{h} \rightarrow gl(W)$ is faithful, then the above pair $(\mathfrak{h} \times W, \mathfrak{h})$ is effective and conversely, for an effective pair $(\mathfrak{g}, \mathfrak{h})$ the kernel of $Ad_{\mathfrak{h}, \mathfrak{g}} : \mathfrak{h} \rightarrow gl(\mathfrak{g})$ is $Z(\mathfrak{g}) \cap \mathfrak{h} = \{0\}$. Observe that $ord(\mathfrak{h} \times W, \mathfrak{h}) = 1$. Recall that a Lie algebra $\mathfrak{g} \supset \mathfrak{h}$ with $(\mathfrak{g}, \mathfrak{h})$ effective defines a "flag" inside $Nil(\mathfrak{h})$ according to (13). The next proposition therefore gives a far reaching generalization of the Ado's theorem.

Proposition 47 Let \mathfrak{h} be a (finite dimensional) Lie algebra with a flag \mathcal{F} inside $Nil(\mathfrak{h})$. Then there exists a Lie algebra $\mathfrak{g} \supset \mathfrak{h}$ such that $(\mathfrak{g}, \mathfrak{h})$ is effective and defines \mathcal{F} . In particular, there exists an effective pair $(\mathfrak{g}, \mathfrak{h})$ with $ord(\mathfrak{g}, \mathfrak{h}) = \dim Nil(\mathfrak{h})$.

Finally, let $Iso_k(\mathfrak{h})$ denote the set of the above isomorphism classes with $\dim \mathfrak{g} - \dim \mathfrak{h} = k \geq 1$. This set is obtained from the set of all representations of \mathfrak{h} of rank k but the concept of isomorphism is more stringent than the usual

concept of isomorphism of two representations. For the Lie algebras \mathfrak{g}_i , $i = 1, 2, 3$ in Example 3, for instance, the representations $\rho_i : \mathfrak{o}(n) \rightarrow \mathfrak{g}_i/\mathfrak{o}(n)$ are isomorphic whereas the pairs $(\mathfrak{g}_i, \mathfrak{o}(n))$ are mutually nonisomorphic. Clearly the cardinality $\sharp Iso_k(\mathfrak{h})$ is equal to the uniformization number $\sharp(\mathfrak{h}, k)$ defined in Section 2 if we assume effectiveness.

12 Appendix D. The adjoint representation

Let $(M, \varepsilon\mathcal{G}_k)$ be a locally solvable *phg* and recall the presheaf $\mathfrak{g}(U)$ whose sections are the local solutions of $\varepsilon\mathfrak{G}_k \rightarrow M$ on U . We choose some $e \in U$ and define

$$\begin{aligned} j_k(\cdot)^{e,e} &: \mathfrak{g}(U) \rightarrow (\mathfrak{G}_k)^e \\ &: \xi \rightarrow j_k(\xi)^{e,e} \end{aligned} \quad (98)$$

Now $(\mathfrak{G}_k)^e$ is a Lie algebra endowed with the algebraic bracket (19) and (98) is an isomorphism of Lie algebras for sufficiently small and simply connected U . Let $H^*(M, \mathfrak{g})$ denote the cohomology groups of M with coefficients in the sheaf \mathfrak{g} . Since \mathfrak{g} is the kernel of the first operator in (45) and partition of unity applies to sections of the spaces in (45), (45) is a fine resolution of the sheaf \mathfrak{g} and therefore computes $H^*(M, \mathfrak{g})$. For simplicity of notation, we denote the Lie algebra $(\mathfrak{G}_k)^e$ by \mathfrak{g}_e and let $H^*(\mathfrak{g}_e, \mathfrak{g}_e)$ denote the deformation cohomology of \mathfrak{g}_e .

Proposition 48 *If M is compact and simply connected, then*

$$H^*(M, \mathfrak{g}) \simeq H^*(\mathfrak{g}_e, \mathfrak{g}_e) \quad (99)$$

For simplicity we now assume that the pseudogroup $(\widetilde{M}, \varepsilon\mathcal{G}_k)$ is globalizable to G so that any initial condition of $(\mathfrak{G}_k)^e$ in (98) comes from a global section of $\mathfrak{g}(M)$ = the Lie algebra of infinitesimal generators of the action of G on M . Now G acts on $\Gamma\mathfrak{G}_k$ = the space of the global sections of the algebroid $\mathfrak{G}_k \rightarrow M$ by

$$(g \cdot s)(x) \stackrel{def}{=} \varepsilon j_k(g)_*^{x, g(x)} s(g^{-1}(x)) \quad (100)$$

Thus $\Gamma\mathfrak{G}_k$ is an infinite dimensional representation space for G and $\mathfrak{g}(M) \subset \Gamma\mathfrak{G}_k$ is a stable and finite dimensional subspace. The adjoint representation of G on $\mathfrak{g}(M)$ localizes as follows: We identify $\xi \in \mathfrak{g}(M)$ with $j_k(\xi)^{e,e}$ by (98). Now $j_k(\xi)^{g(e), g(e)}$ determines some $\zeta \in \mathfrak{g}(M)$ and $Ad(g)^e$ is the map $j_k(\xi)^{e,e} \rightarrow j_k(\zeta)^{e,e}$.

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